# SELF-SIMILAR ACTIONS OF GROUPS ON GRAPHS

LACHLAN MACDONALD

## 1. Introduction

The properties of self-similar actions of groups on sets have been investigated by Laca-Raeburn-Ramagge-Whittaker through their associated Toeplitz and Cuntz-Pimsner algebras [3]. In this project we attempt to generalise the definition of a self-similar action to a group $G$ on a finite directed graph $E$. We investigate the operator algebraic properties of the resulting structure and deduce that each self-similar action of a group on a finite directed graph has a universal $C^*$-algebra generated by an isomorphic copy of the Toeplitz graph $C^*$-algebra of $E$ and an isomorphic copy of the group $C^*$-algebra of $G$.

## 2. Self-similar actions

We begin our investigation by defining a self-similar action of a group on a graph.

**Definition 2.0.1.** *Let $E$ be a finite, directed graph with surjective range map and let $G$ be a group. Denote by $E^*$ the set of all finite paths in $E$. A self-similar action $(G, E)$ is a faithful action of $G$ on $E^*$ such that*

$$g \cdot r(f) = r(g \cdot f)$$

*for all $g \in G$ and $f \in E^1$ and for every $f \in E^1$ and $g \in G$, $g \cdot f \in E^1$ and there exists unique $g|_f \in G$ with*

$$(g \cdot f\mu) = (g \cdot f)(g|_f \cdot \mu)$$

*for all $\mu \in E^*$ with $r(\mu) = s(f)$*

**Remark 2.0.1.** *For $g \in G$ and $f \in E^1$, $g \cdot f = g \cdot r(f)f = (g \cdot r(f))(g \cdot f)$ so that $g|_v := g$ satisfies $g \cdot vf = (g \cdot v)(g|_v \cdot f)$ whenever $v = r(f)$. There is generally no group element which uniquely satisfies this.*

**Lemma 2.0.1.** *Let $(G, E)$ be a self-similar action. Then*

1. *For all $g \in G$ and $\mu \in E^*$, $g \cdot \mu \in E^0$ if and only if $\mu \in E^0$*
2. *For every $v \in E^0$ and $g \in G$, $f \mapsto g \cdot f$ is a bijection from $r^{-1}(v)$ to $r^{-1}(g \cdot v)$.*
3. *For all $g \in G$ and $N \in \mathbb{N}\{0\}$, $\mu \in E^N \Rightarrow g \cdot \mu \in E^N$.*

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(4) *For each* $N \in \mathbb{N} \setminus \{0\}$ *and* $(g, \nu) \in G \times E^N$, *there exists unique* $g|_\nu \in G$ *satisfying*

$$g \cdot (\nu\mu) = (g \cdot \nu)(g|_\nu \cdot \mu)$$

*for all* $\mu \in E^*$ *with* $r(\mu) = s(\nu)$.

(5) *For* $g, h \in G$ *and* $\mu, \nu \in E^*$ *with* $r(\mu) = s(\nu)$, *we have*

(a)

$$g|_{\nu\mu} = (g|_\nu)|_\mu$$

(b)

$$gh|_\nu = g|_{h\cdot\nu} h|_\nu$$

(c)

$$g|_\nu^{-1} = g^{-1}|_{g\cdot\nu}$$

(6) *For every* $g \in G$ *and* $N \in \mathbb{N}$, $g : E^N \to E^N$ *is bijective.*

*Proof.* (1) Fix $\mu \in E^*$ and $g \in G$. For the forward direction, suppose that $g \cdot \mu \in E^0$. Since $r$ is surjective, $u := g \cdot \mu$ is the range of some edge $f$. Now $\mu = g^{-1} \cdot g \cdot \mu = g^{-1} \cdot u = g^{-1} \cdot r(f) = r(g^{-1} \cdot f) \in E^0$. For the reverse direction, if $\mu \in E^0$, $\mu = r(h)$ for some edge $h$, and $g \cdot \mu = g \cdot r(h) = r(g \cdot h) \in E^0$.

(2) Fix $v \in E^0$ and $g \in G$. For $f, h \in r^{-1}(v)$ such that $g \cdot f = g \cdot h$, we have $f = g^{-1} \cdot g \cdot f = g^{-1} \cdot g \cdot h = h$, hence $f \mapsto g \cdot f$ is injective. For $f \in r^{-1}(g \cdot v)$, $f = g \cdot g^{-1} \cdot f$ and $r(f) = g \cdot v \Rightarrow r(g^{-1} \cdot f) = v$, so that $f \mapsto g \cdot f$ is surjective.

(3) Fix $g \in G$ and $N \in \mathbb{N}\{0\}$. If $N = 0$, then $g \cdot \mu \in E^0$ by (1). If $N = 1$, then $g \cdot \mu \in E^1$ by definition of a self-similar action. Now suppose that $g \cdot \mu \in E^{N-1}$ whenever $\mu^{N-1}$. Then for $\mu \in E^N$,

$$g \cdot \mu = (g \cdot \mu_1)(g|_{\mu_1} \cdot \mu_2 ... \mu_N) \in E^N$$

(4) Fix $N \in \mathbb{N} \setminus \{0\}$ and $(g, \nu) \in G \times E^N$. If $r(\mu) = s(\nu)$, then

$$g \cdot \nu\mu = (g \cdot \nu)...(g|_{\nu_1 ...}|_{\nu_N} \cdot \mu)$$

so that $g|_\nu := g|_{\nu_1}|_{\nu_2 ...}|_{\nu_N}$ has the desired property. For uniqueness, observe that when $\mu \in E^1$, we know that $g|_\mu$ uniquely satisfies the desired property by definition of a self-similar action. Suppose now that whenever $\alpha \in E^{N-1}$ and $g \cdot (\alpha\mu) = (g \cdot \alpha)(h \cdot \mu)$ for all $\mu$ with $r(\mu) = s(\alpha)$, we have $h = g|_\alpha$. Now fix $\nu \in E^N$, and suppose $g \cdot (\nu\mu) = (g \cdot \nu)(h \cdot \mu)$ for all $\mu$ with $r(\mu) = s(\nu)$. We must show that $h = g|_\nu$. We have

$$\begin{aligned} g \cdot \nu\mu &= (g \cdot \nu_1)(g|_{\nu_1} \cdot (\nu_2 ... \nu_N)\mu) \\ &= (g \cdot \nu_1)(g|_{\nu_1} \cdot (\nu_2 ... \nu_N))(g|_{\nu_1}|_{\nu_2 ... \nu_N} \cdot \mu) \end{aligned}$$

and

$$(g \cdot \nu)(h \cdot \nu) = (g \cdot \nu_1)(g|_{\nu_1} \cdot (\nu_2 ... \nu_N))(h \cdot \mu)$$

so

$$(g \cdot \nu_1)(g|_{\nu_1} \cdot (\nu_2 ... \nu_N))(g|_{\nu_1}|_{\nu_2 ... \nu_N} \cdot \mu) = (g \cdot \nu_1)(g|_{\nu_1} \cdot (\nu_2 ... \nu_N))(h \cdot \mu)$$

for all $\mu$ with $r(\mu) = s(\nu)$ gives $h = g|_\nu$.

(5) Fix $g, h \in G$ and $\mu, \nu \in E^*$ with $r(\mu) = s(\nu)$.

(a) If $|\nu\mu| = 0$, then the assertion reduces to $g = g$. Suppose $|\nu\mu| > 0$. If $r(\lambda) = s(\mu)$, then

$$\begin{aligned} g \cdot \nu\mu\lambda &= (g \cdot \nu)(g|_\nu \cdot \mu\lambda) \\ &= (g \cdot \nu)(g|_\nu \cdot \mu)(g|_\nu|_\mu \cdot \lambda) \\ &= (g \cdot \nu\mu)(g|_\nu|_\mu \cdot \lambda) \end{aligned}$$

and

$$g \cdot \nu\mu\lambda = (g \cdot \nu\mu)(g|_{\nu\mu} \cdot \lambda)$$

so that $g|_{\nu\mu} = g|_\nu|_\mu$ by (4).

(b) If $|\nu| = 0$, then assertion reduces to $g = g$. Suppose $|\nu| > 0$. If $r(\lambda) = s(\mu)$, then

$$\begin{aligned} g \cdot h \cdot \nu\lambda &= g \cdot (h \cdot \nu)(h|_\nu \cdot \lambda) \\ &= (g \cdot (h \cdot \nu))(g|_{h\cdot\nu} \cdot h|_\nu \cdot \lambda) \\ &= (gh \cdot \nu)(g|_{h\cdot\nu} h|_\nu \cdot \lambda) \\ &= (gh \cdot \nu\lambda) \\ &= (gh \cdot \nu)(gh|_\nu \cdot \lambda) \end{aligned}$$

so that $gh|_\nu = g|_{h\cdot\nu} h|_\nu$ by (4).

(c)

$$\begin{aligned} \nu\lambda &= g^{-1} \cdot g \cdot \nu\lambda \\ &= g^{-1} \cdot (g \cdot \nu)(g|_\nu \lambda) \\ &= \nu(g^{-1}|_{g\cdot\nu} g|_\nu \lambda) \end{aligned}$$

so that $g|_\nu^{-1} = g^{-1}|_{g\cdot\nu}$ by uniqueness of inverses in the group and (4).

(6) We showed in (2) that the map is injective. Since $E^N$ is finite, (3) and the pigeonhole principle imply that the map is bijective. □

## 3. Operator-algebraic properties

Having defined a self-similar action of a group on a graph and proved some of the characteristics of such an action, we now investigate the properties of such structures in the context of $C^*$-algebras.

3.1. **Representations.** Representations are what we will use to embed the structure of self-similar actions in a $C^*$-algebra.

**Definition 3.1.1.** *Let $(G, E)$ be a self-similar action. A Toeplitz representation $(v, t)$ of $(G, E)$ in a unitary $C^*$-algebra $B$ is a pair of maps $v : G \to B$ and $t : E^* \to B$ with the following properties:*

(1) *$v$ is a unitary representation of $G$ in $B$.*

(2) *The set $\{t_\mu : \mu \in E^*\}$ has*

(a) $\{t_v : v \in E^0\}$ *are mutually orthogonal projections, with* $\sum_{v \in E^0} t_v = 1_B$,
(b) $t_\mu t_\nu = t_{\mu\nu}$ *whenever* $s(\mu) = r(\nu)$,
(c) $t_\mu^* t_\mu = t_{s(\mu)}$ *for all* $\mu \in E^*$,
(d) *For all* $n \in \mathbb{N}$ *and* $v \in E^0$,

$$\sum_{\substack{r(\mu)=v \\ |\mu|=n}} t_\mu t_\mu^* \leq t_v$$

(3) $v_g t_\mu = t_{g\cdot\mu} v_{g|_\mu}$ *for all* $g \in G$ *and* $\mu \in E^*$.

**Remark 3.1.1.** *The definition of a Toeplitz representation of a self-similar action encodes the information contained about the group and graph. Condition (1) says a quotient copy of the group is contained in the unitaries of the enveloping* $C^*$*-algebra. Condition (2) expresses the same properties carried by the Toeplitz algebra of the graph* $E$*. Condition (3) expresses the structure of the action itself.*

**Lemma 3.1.1.** *Let* $(G, E)$ *be a self-similar action and let* $(v, t)$ *be a Toeplitz representation of* $(G, E)$ *in a unital* $C^*$*-algebra* $B$*. If* $g \in G$ *and* $\mu, \nu \in E^*$ *satisfy* $t_\mu v_g t_\nu \neq 0$*, then* $g \cdot s(\nu) = s(\mu)$*. Suppose that* $g, h \in G$ *and* $\mu, \nu, \sigma, \tau \in E^*$ *satisfy* $g \cdot s(\nu) = s(\mu)$ *and* $h \cdot s(\tau) = s(\sigma)$*. Then*

$$t_\mu v_g t_\nu^* t_\sigma v_h t_\tau^* = \begin{cases} t_\mu v_{gh|_{h^{-1}\cdot\nu'}} t^*_{\tau(h^{-1}\cdot\nu')} & \text{if } \nu = \sigma\nu' \\ t_{\mu(g\cdot\sigma')} v_{g|_{\sigma'}h} t_\tau^* & \text{if } \sigma = \nu\sigma' \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Fix $g, h \in G$ and $\mu, \nu, \tau, \sigma \in E^*$.

$$\begin{aligned} t_\mu v_g t_\nu^* \neq 0 &\Rightarrow t_\mu t_{s(\mu)} v_g t_\nu^* \neq 0 \\ &\Rightarrow t_\mu v_g t_{g^{-1}\cdot s(\mu)} t_\nu^* \neq 0 \\ &\Rightarrow t_\mu v_g (t_\nu t_{g^{-1}\cdot s(\mu)})^* \neq 0 \\ &\Rightarrow s(\nu) = g^{-1} \cdot s(\mu) \\ &\Rightarrow g \cdot s(\nu) = s(\mu) \end{aligned} \tag{1}$$

Now, suppose $\nu = \sigma\nu'$ for some $\nu'$. Then

$$\begin{aligned}
t_\mu v_g t_\nu^* t_\sigma v_h t_\tau^* &= t_\mu v_g t_{\sigma\nu'}^* t_\sigma v_h t_\tau^* \\
&= t_\mu v_g t_{\nu'}^* t_\sigma^* t_\sigma v_h t_\tau^* \\
&= t_\mu v_g t_{\nu'}^* t_{s(\sigma)} v_h t_\tau^* \\
&= (t_\tau v_h^* t_{s(\sigma)} t_{\nu'} v_g^* t_{\nu'}^*)^* \\
&= (t_\tau v_h^* t_{\nu'} v_g^* t_\mu^*)^* \qquad \text{since } s(\sigma) = r(\mu) \\
&= t_\mu v_g t_{\nu'}^* v_h t_\tau^* \\
&= t_\mu v_g (v_{h^{-1}} t_{\nu'})^* t_\tau^* \\
&= t_\mu v_g (t_{h^{-1}\cdot\nu'} v_{h^{-1}|_{\nu'}})^* t_\tau^* \\
&= t_\mu v_g v_{h^{-1}|_{\nu'}}^{-1} (t_\tau t_{h^{-1}\cdot\nu'})^* \\
&= t_\mu v_{gh|_{h^{-1}\cdot\nu'}} t_{\tau(h^{-1}\cdot\nu')}^* \qquad (2)
\end{aligned}$$

Where the last line is true since (1) implies that $s(\tau) = h^{-1} \cdot r(\nu') = r(h^{-1} \cdot \nu')$ because $r(\nu') = s(\sigma)$.

Now suppose $\sigma = \nu\sigma'$. Then

$$\begin{aligned}
t_\mu v_g t_\nu^* t_\sigma v_h t_\tau^* &= t_\mu v_g t_\nu^* t_{\nu\sigma'} v_h t_\tau^* \\
&= t_\mu v_g t_{s(\nu)} t_{\sigma'} v_h t_\tau^* \\
&= t_\mu v_g t_{\sigma'} v_h t_\tau^* \qquad \text{since } s(\nu) = r(\sigma') \\
&= t_\mu t_{g\cdot\sigma'} v_{g|_{\sigma'}} v_h t_\tau^* \\
&= t_{\mu(g\cdot\sigma')} v_{g|_{\sigma'}h} t_\tau^* \qquad (3)
\end{aligned}$$

Where the last line is true since (1) implies that $s(\mu) = g \cdot s(\nu) = g \cdot r(\sigma') = r(g \cdot \sigma')$. If neither of the above hypotheses hold, then we have $r(\mu) \neq r(\sigma)$ and so the product $t_\nu^* t_\sigma = t_\nu^* t_{r(\nu)} t_{r(\sigma)} t_\sigma$ is zero, as $t_{r(\nu)}$ and $t_{r(\sigma)}$ are then mutually orthogonal projections. This, combined with equations (2) and (3), gives the result. □

**Lemma 3.1.2.** *Let $(G, E)$ be a self-similar action and let $(v, t)$ be a Toeplitz representation of $(G, E)$ in a unital $C^*$-algebra $B$. Then*

$$C^*(\{v_g\} \cup \{t_\mu\}) = \overline{span}\{t_\mu v_g t_\nu^* : g \in G, \mu, \nu \in E^*\}$$

*Proof.* Define $K := span\{t_\mu v_g t_\nu^* : g \in G, \mu, \nu \in E^*\}$. For the reverse containment, we see that $K \subset C^*(\{v_g\} \cup \{t_\mu\})$. Since $C^*(\{v_g\} \cup \{t_\mu\})$ is a $C^*$-algebra, it is complete, so $\overline{K} = \overline{span}\{t_\mu v_g t_\nu^* : g \in G, \mu, \nu \in E^*\} \subset C^*(\{v_g\} \cup \{t_\mu\})$. For the reverse containment, we see that $K$ contains the generating set $\{v_g\} \cup \{t_\mu\}$, for fix $g \in G$ and $\mu \in E^*$. Then

$$\begin{aligned} v_g &= \sum_{u' \in E^0} t_{u'} v_g \\ &= \sum_{u \in E^0} \sum_{u' \in E^0} t_{u'} v_g t_u \\ &= \sum_{u \in E^0} \sum_{u' \in E^0} t_{u'} v_g t_u^* \text{ each } t_u \text{ is a projection} \\ &\in K \end{aligned}$$

And,

$$\begin{aligned} t_\mu &= t_\mu 1 \\ &= \sum_{u \in E^0} t_\mu 1 t_u \\ &= \sum_{u \in E^0} t_\mu 1 t_u^* \quad \text{as each } t_u \text{ is a projection} \\ &\in K \end{aligned}$$

Further, $K$ is closed under involution in $B$, and is closed under multiplication as proven in the previous lemma. Now $\overline{K} = \overline{span}\{t_\mu v_g t_\nu^* : g \in G, \mu, \nu \in E^*\}$ is a closed *-subalgebra of $B$ containing the generating elements of $C^*(\{v_g\} \cup \{t_\mu\})$, hence $C^*(\{v_g\} \cup \{t_\mu\}) \subset \overline{span}\{t_\mu v_g t_\nu^* : g \in G, \mu, \nu \in E^*\}$ and we have established reverse containment. □

### 3.2. **The Universal $C^*$-algebra.**

**Theorem 3.2.1.** *Let $(G, E)$ be a self-similar action. There is a $C^*$-algebra $\mathcal{T}C^*(G, E)$ generated by a Toeplitz representation $(u, s)$ which is universal in the sense that for any other Toeplitz representation $(v, t)$ in a $C^*$-algebra $B$, there is a homomorphism $\pi_{v,t} : \mathcal{T}C^*(G, E) \to B$ satisfying $\pi_{v,t}(u_g) = v_g$ and $\pi_{v,t}(s_\mu) = t_\mu$. We call this universal $C^*$-algebra the Toeplitz algebra of $(G, E)$.*

*Proof.* Suppose $(G, E)$ is a self-similar action. Let $A$ be the formal vector space spanned by finite linear combinations of the form

$$\sum_{(\mu,g,\nu) \in F} a_{\mu,g,\nu} \theta_{\mu,g,\nu} \text{ where } F \subset \{(\mu, g, \nu) : g \cdot s(\nu) = s(\mu)\}$$

Define a bilinear multiplication on $A$ by

$$\theta_{\mu,g,\nu} \cdot \theta_{\sigma,h,\tau} = \begin{cases} \theta_{\mu, gh|_{h^{-1}\cdot\nu'}, \tau(h^{-1}\cdot\nu')} & \text{if } \nu = \sigma\nu' \\ \theta_{\mu(g\cdot\sigma'), g|_{\sigma'} h, \tau} & \text{if } \sigma = \nu\sigma' \\ 0 & \text{otherwise} \end{cases}$$

And then $\cdot$ extends to a multiplication on $A$. Define an involution on $A$ via

$$\left(\sum_F a_{\mu,g,\nu} \theta_{\mu,g,\nu}\right)^* := \sum_F \overline{a_{\mu,g,\nu}} \theta_{\nu,g,\mu}$$

Then $A$ is a complex *-algebra. Given a Toeplitz representation $(v,t)$ of $(G,E)$ in a $C^*$-algebra $B$, there is a *-homomorphism $\pi^0_{v,t} : A \to C^*(v,t)$ defined by $\pi^0_{v,t}(\theta_{\mu,g,\nu}) := t_\mu u_g t^*_\nu$, since $C^*(v,t)$ is precisely the closed span of elements of this form, as proven in Lemma 3.1.2. For $a \in A$, define

$$N(a) := \sup\{\|\pi^0_{v,t}(a)\| : (v,t) \text{ is a Toeplitz representation of } (G,E)\}$$

We note that for each $a \in A$, $N(a)$ is finite:

$$\begin{aligned} N(a) &= \sup\{\|\sum_F a_{\mu,g,\nu} t_\mu v_g t^*_\nu\| : (v,t) \text{ is a Toeplitz representation}\} \\ &\leq \sup\{\sum_F \|a_{\mu,g,\nu} t_\mu v_g t^*_\nu\| : (v,t) \text{ is a Toeplitz representation}\} \qquad (4) \end{aligned}$$

Now,

$$\begin{aligned} \|t_\mu v_g t^*_\nu\|^2 &= \|t_\mu v_g t^*_\nu t_\nu v_{g^{-1}} t^*_\mu\| \\ &= \|t_\mu v_g t_{s(\nu)} v_{g^{-1}} t^*_\mu\| \\ &= \|t_\mu t_{g\cdot s(\nu)} v_g v_{g^{-1}} t^*_\mu\| \\ &= \|t_\mu t^*_\mu\| \text{ since } g \cdot s(\nu) = s(\mu) \\ &= \|t_\mu\|^2 \\ &= \|t^*_\mu t_\mu\| \\ &= \|t_{s(\mu)}\| \\ &= 1 \qquad (5) \end{aligned}$$

Where the last line is true because $t_{s(\mu)}$ is a projection. Thus, combining (4) with (5)W gives

$$N(a) \leq \sup\{\sum_F |a_{\mu,g,\nu}|\} = \sum_F |a_{\mu,g,\nu}|$$

Which is a finite sum of finite positive numbers, hence finite. $N$ descends to a $C^*$-norm on the *-algebra $\overline{A/ker(N)}$: for $a + ker(N), b + ker(N) \in \overline{A/ker(N)}$, we have

$$\begin{aligned} \|(a+ker(N))^*(a+ker(N)\| &= \|a^*a + ker(N)\| \\ &= N(a^*a) \\ &= \sup_{(v,t)}\{\|\pi^0_{v,t}(a^*a)\|\} \\ &= \sup_{(v,t)}\{\|\pi^0_{v,t}(a)\|^2\} \\ &= \sup_{(v,t)}\{\|\pi^0_{v,t}(a)\|\}^2 \\ &= N(a)^2 \\ &= \|a + ker(N)\|^2 \end{aligned}$$

and

$$\begin{aligned}
\|(a+ker(N))(b+ker(N))\| &= \|ab+ker(N)\| \\
&= \sup_{(v,t)}\{\|\pi^0_{v,t}(ab)\|\} \\
&\leq \sup_{(v,t)}\{\|\pi^0_{v,t}(a)\|\|\pi^0_{v,t}(b)\|\} \\
&\leq \sup_{(v,t)}\{\|\pi^0_{v,t}(a)\|\}\sup_{(v,t)}\{\|\pi^0_{v,t}(a)\|\} \\
&= N(a)N(b) \\
&= \|(a+ker(N))\|\|(b+ker(N))\|
\end{aligned}$$

so that $\overline{A/ker(N)}$ is a $C^*$-algebra. We now define families $\{s_\mu : \mu \in E^*\}$ and $\{u_g : g \in G\}$ ias$\overline{A/ker(N)}$

$$s_\mu := \theta_{\mu,e,s(\mu)}$$

$$u_g := \sum_{v\in E^0} \theta_{v,g,g^{-1}\cdot v}$$

These families induce a Toeplitz representation $(u,s)$ of $(G,E)$ in $\overline{A/ker(N)}$ (note that it suffices to check the Toeplitz representation axioms in $A$, as the quotient map is a homomorphism):

(1) $u$ is a unitary map: The element $\sum_{v\in E^0}\theta_{v,e,v}$ is a multiplicative identity. For $\theta_{\mu,g,\nu} \in A$, we have

$$\begin{aligned}
(\sum_{v\in E^0}\theta_{v,e,v})\theta_{\mu,g,\nu} &= \sum_{v\in E^0}\theta_{v,e,v}\theta_{\mu,g,\nu} \\
&= \theta_{v\mu,eg,\nu} \text{ where } v = r(\mu) \\
&= \theta_{\mu,g,\nu}
\end{aligned}$$

and

$$\begin{aligned}
\theta_{\mu,g,\nu}(\sum_{v\in E^0}\theta_{v,e,v}) &= \sum_{v\in E^0}\theta_{\mu,g,\nu}\theta_{v,e,v} \\
&= \theta_{\mu,g,v\nu} \text{ where } v = r(\nu) \\
&= \theta_{\mu,g,\nu}
\end{aligned}$$

Now for $g \in G$, we have

$$\begin{aligned}
u_{g^{-1}} &= \sum_{v\in E^0}\theta_{v,g^{-1},g\cdot v} \\
&= \sum_{v\in E^0}\theta_{g^{-1}\cdot v,g^{-1},v} \\
&= u_g^*
\end{aligned}$$

and

$$\begin{aligned} u_{g^{-1}}u_g &= (\sum_{v\in E^0}\theta_{v,g^{-1},g\cdot v})(\sum_{v\in E^0}\theta_{v,g,g^{-1}\cdot v}) \\ &= \sum_{v\in E^0}\sum_{v'\in E^0}\theta_{v,g^{-1},g\cdot v}\cdot\theta_{v',g,g^{-1}\cdot v'} \\ &= \sum_{v\in E^0}\theta_{v,g^{-1},g\cdot v}\theta_{g\cdot v,g,g^{-1}\cdot(g\cdot v)} \\ &= \sum_{v\in E^0}\theta_{v,g^{-1}g|_{g^{-1}\cdot(g)},g^{-1}\cdot(g\cdot v)(g^{-1}\cdot(g\cdot v))} \\ &= \sum_{v\in E^0}\theta_{v,e,v} \end{aligned}$$

and since $g\in G$ was arbitrary, the above argument works for the reverse multiplication by replacing $g$ with $g^{-1}$. Finally, for $g,h\in G$,

$$\begin{aligned} u_g u_h &= (\sum_{v\in E^0}\theta_{v,g,g^{-1}\cdot v})(\sum_{v\in E^0}\theta_{v,h,h^{-1}\cdot v}) \\ &= \sum_{v\in E^0}\sum_{v'\in E^0}\theta_{v,g,g^{-1}\cdot v}\cdot\theta_{v',h,h^{-1}\cdot v'} \\ &= \sum_{v\in E^0}\theta_{v(g\cdot(g^{-1}\cdot v)),g|_{g^{-1}\cdot v}h,h^{-1}\cdot(g^{-1}\cdot v)} \\ &= \sum_{v\in E^0}\theta_{v,gh,(gh)^{-1}\cdot v} \\ &= u_{gh} \end{aligned}$$

So that $u$ is a unitary homomorphism of $G$ into $\mathcal{U}(\overline{A/ker(N)})$, as required.

(2) Considering the family $\{s_\mu : \mu\in E^*\}$, we have:

(a) For $\mu,\nu\in E^*$,

$$\begin{aligned} s_\mu s_\nu &= \theta_{\mu,e,s(\mu)}\cdot\theta_{\nu,e,s(\nu)} \\ &= \delta_{(\nu),s(\mu)}\theta_{\mu\nu,e,s(\nu)} \\ &= s_{\mu\nu}\text{ if } r(\nu)=s(\mu) \end{aligned}$$

(b) For $v,u\in E^0$,

$$\begin{aligned} s_v s_u &= \theta_{v,e,v}\theta_{u,e,u} \\ &= \delta_{v,u}\theta_{v,e,v} \end{aligned}$$

with the property that $\sum_{v\in E^0}s_v=\sum_{v\in E^0}\theta_{v,e,v}=1$.

(c) For $\mu\in E^*$,

$$\begin{aligned} s_\mu^* s_\mu &= \theta_{s(\mu),e,\mu}\theta_{\mu,e,s(\mu)} \\ &= \theta_{s(\mu),e,s(\mu)} \\ &= s_{s(\mu)} \end{aligned}$$

(d) For $n \in \mathbb{N}$, $v \in E^0$, we see that $\sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^* = \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu}$ is self-adjoint:

$$\begin{aligned}(\sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu})^* &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta^*_{\mu,e,\mu}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu}\end{aligned}$$

and that $\sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^* = (\sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^*)^2$ since

$$\begin{aligned}(\sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu})^2 &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \sum_{\substack{r(\nu)=v\\|\nu|=n}} \theta_{\mu,e,\mu}\theta_{\nu,e,\nu}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu s(\mu),e,\mu}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu}\end{aligned}$$

So $\sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^*$ is a projection for each $v \in E^0$ and $n \in \mathbb{N}$.
Now for $v \in E^0$,

$$\begin{aligned}(\sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^*) s_v &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu} \cdot \theta_{v,e,v}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,v(e\cdot\mu)}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} \theta_{\mu,e,\mu}\\ &= \sum_{\substack{r(\mu)=v\\|\mu|=n}} s_\mu s_\mu^*\end{aligned}$$

and

$$
\begin{aligned}
s_v(\sum_{\substack{r(\mu)=v \\ |\mu|=n}} s_\mu s_\mu^*) &= \sum_{\substack{r(\mu)=v \\ |\mu|=n}} \theta_{v,e,v} \cdot \theta_{\mu,e,\mu} \\
&= \sum_{\substack{r(\mu)=v \\ |\mu|=n}} \theta_{v(e\cdot\mu),e,\mu} \\
&= \sum_{\substack{r(\mu)=v \\ |\mu|=n}} \theta_{\mu,e,\mu} \\
&= \sum_{\substack{r(\mu)=v \\ |\mu|=n}} s_\mu s_\mu^*
\end{aligned}
$$

so by [4],

$$
\sum_{\substack{r(\mu)=v \\ |\mu|=n}} s_\mu s_\mu^* \leq s_v
$$

(3) Finally, for $g \in G$ and $\mu \in E^*$,

$$
\begin{aligned}
u_g s_\mu &= \sum_{v\in E^0} \theta_{v,g,g^{-1}\cdot v} \cdot \theta_{\mu,e,s(\mu)} \\
&= \sum_{v\in E^0} \theta_{g\cdot v,g,v} \theta_{\mu,e,s(\mu)} \\
&= \theta_{g\cdot r(\mu),g,r(\mu)} \cdot \theta_{\mu,e,s(\mu)} \\
&= \theta_{g\cdot r(\mu)(g\cdot\mu),g|_\mu e,s(\mu)} \\
&= \theta_{g\cdot\mu,g|_\mu,s(\mu)} \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
s_{g\cdot\mu} u_{g|_\mu} &= \theta_{g\cdot\mu,e,s(g\cdot\mu)} \sum_{v\in E^0} \theta_{g|_\mu\cdot v,g|_\mu,v} \\
&= \sum_{v\in E^0} \theta_{g\cdot\mu,e,s(g\cdot\mu)} \cdot \theta_{g|_\mu\cdot v,g|_\mu,v} \\
&= \theta_{g\cdot\mu,e,s(g\cdot\mu)} \cdot \theta_{g|_\mu\cdot s(\mu),g|_\mu,s(\mu)} \\
&= \theta_{g\cdot\mu,eg|_\mu|_{(g|_\mu^{-1}\cdot s(g\cdot\mu))},s(\mu)(g|_\mu^{-1}\cdot s(g\cdot\mu))} \\
&= \theta_{g\cdot\mu,g|_\mu,s_\mu} \\
&= u_g s_\mu
\end{aligned}
$$

So $(u, s)$ as defined is a Toeplitz representation. We now define $\mathcal{T}C^*(G, E) := \overline{A/ker(N)}$. For the universal property, let $(v, t)$ be a Toeplitz representation of $(G, E)$ in a $C^*$-algebra $B$. Define $\pi_{v,t} : \mathcal{T}C^*(G, E) \to B$ by $\pi_{v,t}(a + ker(N)) := \pi^0_{v,t}(a)$, and $\pi_{v,t}$ has the desired property. $\square$

**Lemma 3.2.1.** *Let $(G, E)$ be a self similar action. Suppose that $\pi$ is a representation of $\mathcal{T}C^*(G, E)$ on a Hilbert space $\mathcal{H}$ and that $\rho$ is a representation of $C^*(G)$ on a Hilbert space $\mathcal{K}$. Then there is a representation $\pi$ of $\mathcal{T}C^*(G, E)$ on $\mathcal{H} \otimes \mathcal{K}$ such that*

$$(\pi \times \rho)(u_g) = \pi(u_g) \otimes \rho(i_G(g))$$

*and*

$$(\pi \times \rho)(s_\mu) = \pi(s_\mu) \otimes 1_{\mathcal{K}}$$

*Proof.* Let $(G, E)$ be a self similar action, let $\pi$ be a representation of $\mathcal{T}C^*(G, E)$ on a Hilbert space $\mathcal{H}$ and let $\rho$ be a representation of $C^*(G)$ on a Hilbert space $\mathcal{K}$. We define the linear map $(\pi \times \rho) : \mathcal{T}C^*(G, E) \to \mathcal{H} \otimes \mathcal{K}$ on the spanning elements of $\mathcal{T}C^*(G, E)$ by the formula

$$(\pi \times \rho)(s_\mu u_g s_\nu^*) := \pi(s_\mu u_g s_\nu^*) \otimes \rho(i_G(g))$$

The map is multiplicative and *-preserving because both $\rho$ and $\pi$ are. Now $(\pi \times \rho)$ is a *-homomorphism from $\mathcal{T}C^*(G, E)$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, and is thus a representation of $\mathcal{T}C^*(G, E)$ on $\mathcal{H} \otimes \mathcal{K}$. We see that for $g \in G$ and $\mu \in E^*$, we have

$$\begin{aligned}
(\pi \times \rho)(u_g) &= (\pi \times \rho)\Big(\sum_{v \in E^0} \sum_{v' \in E^0} s_v u_g s_{v'}\Big) \\
&= (\pi \times \rho)\Big(\sum_{v \in E^0} s_v u_g \sum_{v' \in E^0} s_{v'}\Big) \\
&= \pi\Big(\sum_{v \in E^0} s_v u_g \sum_{v' \in E^0} s_{v'}\Big) \otimes \rho(i_G(g)) \\
&= \pi\Big(\sum_{v \in E^0} s_v\Big) \pi(u_g) \pi\Big(\sum_{v' \in E^0} s_{v'}\Big) \otimes \rho(i_G(g)) \\
&= \pi(u_g) \otimes \rho(i_G(g))
\end{aligned}$$

and

$$\begin{aligned}
(\pi \times \rho)(s_\mu) &= (\pi \times \rho)\Big(\sum_{v \in E^0} s_\mu 1 s_v\Big) \\
&= \pi\Big(\sum_{v \in E^0} s_\mu 1 s_v\Big) \otimes \rho(i_G(e)) \\
&= \pi(s_\mu) \otimes 1_{\mathcal{K}}
\end{aligned}$$

So that $(\pi \times \rho)$ is a representation with the desired properties. □

**Theorem 3.2.2.** *Let $(G, E)$ be a self-similar action. Let $(u, s)$ be the universal representation in $\mathcal{T}C^*(G, E)$. Then each $s_\lambda$ is nonzero, and there is an injective homomorphism $\pi_E : \mathcal{T}C^*(E) \to \mathcal{T}C^*(G, E)$ such that $\pi_E(t_\lambda) = s_\lambda$ for all $\lambda \in E^*$. Further, the map $g \mapsto u_g$ induces an injective homomorphism $\iota : C^*(G) \to \mathcal{T}C^*(G, E)$.*

*Proof.* Let $(G, E)$ be a self-similar action and let $(u, s)$ be the universal representation of $(G, E)$. Consider the Hilbert space $\ell^2(E) = \overline{span}\{\delta_\lambda : \lambda \in E^*\}$, where for $\lambda \in E^*$, $\delta_\lambda : E^* \to \mathbb{C}$ is the map defined by

$$\delta_\lambda(\mu) = \begin{cases} 1 \text{ if } \lambda = \mu \\ 0 \text{ otherwise} \end{cases}$$

We construct a Toeplitz representation $(U, S)$ of $(G, E)$ in $\mathcal{B}(\ell^2(E^*))$ via

$$U_g \delta_\mu := \delta_{g \cdot \mu}$$

and

$$S_\lambda \delta_\mu := \begin{cases} \delta_{\lambda\mu} \text{ if } s(\lambda) = r(\mu) \\ 0 \text{ otherwise} \end{cases}$$

$U_g$ and $S_\mu$ are clearly bounded operators for all $g$ and $\mu$, as they map the spanning set of $\ell^2(E^*)$ into itself. It has been shown that the set $\{S_\lambda : \lambda \in E^*\}$ is indeed a Toeplitz-Cuntz-Krieger family of $E$ in $\mathcal{B}(\ell^2(E^*))$ [2][1], so it remains to show that $U$ is a unitary map and that the pair $(U, S)$ satisfies axiom (3) in Definition 3.1.1.

Fix $g \in G$. Then for $\mu, \nu \in E^*$, we have

$$(U_g \delta_\mu | U_g \delta_\nu) = (\delta_{g \cdot \mu} | \delta_{g \cdot \nu})$$

and since $\mu$ and $\nu$ were arbitrary, by faithfulness of the action we have

$$(\delta_{g \cdot \mu} | \delta_{g \cdot \nu}) = \begin{cases} 1 \text{ if } \mu = \nu \\ 0 \text{ otherwise} \end{cases}$$

which is precisely the value of $(\delta_\mu | \delta_\nu)$, and so $U_g$ is an isometry. Furthermore, for $g \in G$ we have $U_{g^{-1}} U_g \delta_\mu = \delta_{g^{-1} \cdot g \cdot \mu} = \delta_\mu$ for all $\mu \in E^*$, so that the $U_g$ are invertible, hence unitary, with $U_g^* = U_{g^{-1}}$.

Now, fix $g \in G$ and $\mu \in E^*$.

$$\begin{aligned} S_{g \cdot \mu} U_{g|_\mu} \delta_\lambda &= S_{g \cdot \mu} \delta_{g|_\mu \cdot \lambda} \\ &= \delta_{(g \cdot \mu)(g|_\mu \cdot \lambda} \\ &= \delta_{g \cdot (\mu\lambda)} \\ &= U_g S_\mu \delta_\lambda \end{aligned}$$

for all $\lambda \in E^*$ with $r(\lambda) = s(\mu)$. For $\lambda$ not satisfying this condition, we have $r(g|_\mu \cdot \lambda) = g|_\mu \cdot r(\lambda) \neq g|_\mu \cdot s(\mu) = s(g \cdot \mu)$, so that

$$S_{g \cdot \mu} U_{g|_\mu} \delta_\lambda = 0 = U_g S_\mu \delta_\lambda$$

and then

$$U_g S_\mu = S_{g \cdot \mu} U_{g|_\mu}$$

for all $g \in G$ and $\mu \in E^*$, and $(U, S)$ is a Toeplitz representation.

The universal property of $\mathcal{T}C^*(G, E)$ now gives a homomorphism $\psi : \mathcal{T}C^*(G, E) \to \mathcal{B}(\ell^2(E^*))$ such that $\psi(u_g) = U_g$ and $\psi(s_\lambda) = S_\lambda$ for all $g \in G$ and $\lambda \in E^*$. In particular, for any $\lambda \in E^*$, $S_\lambda = \psi(s_\lambda) \neq 0$, implying that $s_\lambda \neq 0$.

Further, the family $\{s_\mu : \mu \in E^*\}$ is a Toeplitz-Cuntz-Krieger family in $\mathcal{T}C^*(G,E)$ (contained in condition (2) of the definition of a Toeplitz representation), so that by the universal property of $\mathcal{T}C^*(E)$, the Toeplitz algebra of $E$, there is a homomorphism $\pi_E : \mathcal{T}C^*(E) \to \mathcal{T}C^*(G,E)$ with the property that $\pi_E(t_\lambda) = s_\lambda$ for all $\lambda \in E^*$. Since each $s_\lambda$ is nonzero, $ker(\pi_E) = 0$, so $\pi_E$ is injective.

Finally, to see that $\mathcal{T}C^*(G,E)$ contains a copy of $C^*(G)$, observe that the map $g \mapsto u_g$ from $G$ to $\mathcal{T}C^*(G,E)$ induces a homomorphism $\iota : C^*(G) \to \mathcal{T}C^*(G,E)$ such that $\iota(i_G(g)) = u_g$ for all $g \in G$ by the universal property of $C^*(G)$.

Now let $\pi$ be a representation of $\mathcal{T}C^*(G,E)$ on some Hilbert space $\mathcal{H}$ and let $\rho$ be a faithful representation of $C^*(G)$ on some Hilbert space $\mathcal{K}$. By Lemma 3.2.1, there is a representation $\pi\times\rho$ of $\mathcal{T}C^*(G,E)$ on $\mathcal{H}\otimes\mathcal{K}$ such that $(\pi\times\rho)(u_g) = \pi(u_g)\otimes\rho(i_G(g))$. Now by the universal property of $C^*(G)$, there is an injective homomorphism $\theta : C^*(G) \to \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ induced by the map $g \mapsto (\pi \times \rho)(u_g)$, which is injective because $\rho$ is. Now we have

$$\begin{aligned} \theta(i_G(g)) &= (\pi \times \rho)(u_g) \\ &= (\pi \times \rho)(\iota(i_G(g))) \end{aligned}$$

so that $\iota$ is also injective, as required. □

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