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De Rham's Theorem

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February 28, 2013

Abstract

De Rham's theorem is a classical result implying the existence of an isomorphism between the De Rham and singular cohomology groups of a smooth manifold. In this paper we review basic notions of differential forms, singular simplexes and chain complexes. We then introduce both the de rham and singular cohomologies and show how they are related via Stoke's Theorem. We then present a proof of De Rham's Theorem.

Contents

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1 Introduction

De Rham's Theorem was first proven by Georges De Rham in 1931. The theorem itself was first conjectured by Elie Cartan 2 years earlier, who conjectured that the nth betti number b_n of a smooth manifold M is the maximal number of closed n-forms ω_i such that no linear combination of these forms would be exact(see [3]). De Rham proved that you could use Stoke's theorem to create a dual pair between what is now called the de rham cohomology groups $H^n_{DR}(M)$ and the simplicial homology groups $H_n(M;\mathbb{R})$, with their dimension being equal to the nth betti numbers and thus proving Cartan's conjecture. However at the time, cohomology had not yet been invented and so his original theorem was written only in terms of homology.

In this paper we will give a proof of the more modern De Rhams theorem which is stated in terms of cohomology. That is, for a smooth manifold M, the qth de rham cohoology group $H^q_{DR}(M)$ is isomorphic to the qth singular cohomology group $H^q(M;\mathbb{R})$. In section 2 we will review basic definitions and theorems needed, then in section 3 we will define both the singular and de rham cohomology groups of a smooth manifold M. We will then give important lemmas and theorems needed about these groups before we move onto the mapping induced by stokes theorem and finally give a proof of De Rham's Theorem.

2 Notation and Background

2.1 Differential manifolds and Partitions of unity

De Rham's theorem is a theorem about differential manifolds (also called smooth manifolds or C^∞-manifolds). In this paper we define a differentiable manifold M to consist of two parts, a topology and its differentiable structure. We define its topology to be:

- Hausdorff
- Locally euclidean, each point has some neighbourhood homeomorphic to $\mathbb{R}^d$. d is called the dimension of the manifold
- Second countable. i,e it has a countable basis.

The differentiable structure is a maximal atlas on M such that the transition functions are C^∞-related. This part allows us to do calculus on M. We assume the reader is

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familiar with differentiation on smooth manifolds.

One concept which we will make great use of later are the so called partitions of unity.

Definition 2.1. A *Partition of Unity* on M is a collection $\{\psi_i | i \in I\}$ of C^∞ functions on M such that

- for each $i \in I$, $\psi_i \geq 0$ on M.
- The collection of supports $\{supp\psi_i | i \in I\}$ is locally finite.
- $\sum_{i \in I} \psi_i(p) = 1$ for all $p \in M$.

As it turns out, one of the more important consequences of a smooth manifolds topology is that it is also paracompact. That is, each open cover of M has a locally finite refinement. This is sufficient to always guarantee the existence of partitions of unity.

Theorem 2.1. *For any smooth manifold M and any open cover $\{U_\alpha\}$ on M. There exists a countable partition of unity $\{\psi_i | i \in I\}$ subordinate to the open cover and for each $i \in I$, $supp(\psi_i)$ is compact.*

Proof. See [1, Thm. 1.11] □

Partitions of unity allow us to 'glue' together local functions or properties into global ones and vice versa. We will use these as we will prove De Rham's theorem by breaking the manifold up into smaller more manageable pieces, and use these to glue it back together.

2.2 Differential forms

Differential forms are defined in terms of exterior algebras.

Definition 2.2. We denote the *Exterior Algebra* of V by $\Lambda(V)$. It is graded algebra if we set $\Lambda_k(V)$ as the set of all k-vectors. The multiplication in $\Lambda(V)$ is denoted by $\wedge$ and is call the *wedge* or *exterior product.*

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We will denote the tangent space of M at the point p by T_pM which is the vector space of all linear derivations of germs at p of $C^\infty(M)$ functions. We denote the cotangent space at p as T_pM^* and this is the dual space of the tangent space.

Definition 2.3. Let M be a smooth manifold then

- $\Lambda_k^*(M) = \coprod_{p\in M} \Lambda_k(T_pM^*)$ i.e, the disjoint union of the exterior k-algebra of the cotangent spaces. We call this the exterior k bundle over M.
- $\Lambda^*(M) = \coprod_{p\in M} \Lambda(T_pM^*)$ is called the exterior algebra bundle over M.

These bundles also have natural differential structures. As bundles, they also come equipped with a projection functions π. We call a C^∞-section of $\Lambda^*(M)$ or $\Lambda_k^*(M)$ a (differential) form or (differential) k-form respectively.

We denote the set of all differential k-forms on M as $\Omega^k(M)$. Note that $\Omega^0(M)$ is just the vector space of all $C^\infty(M)$ functions. We can then form the graded algebra $\Omega^*(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus \dots$ over $\mathbb{R}$ of all differential forms on M. The multiplication in this algebra is the product it inherits from the wedge product $\wedge$ of the exterior algebras and is also called the wedge product.

Now, given a 0-form f on a smooth manifold M. For some $m \in M$, it's differential at that point is a linear map $df_m : T_mM \to \mathbb{R}$ and $df : M \to TM^*$ can be considered as a 1-form on M. We can extend this operation to all forms and its called the *Exterior Derivative.*

Theorem 2.2 (Exterior differentiation)**.** *There exists a unique linear operation* $d : \Omega^*(M) \to \Omega^*(M)$ *called the* exterior derivative *that*

- *Takes a k-form to a* $(k+1)$*-form*
- $d \circ d = 0$
- *For* $u \in \Omega^j(M)$ *and* $v \in \Omega^k(M)$, $d(u \wedge v) = d(u) \wedge v + (-1)^j u \wedge d(v)$.
- *Whenever* $f \in \Omega^0(M)$, df *is the differential of* f.

Proof. See [1, Thm. 2.20] □

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The differential of a C^∞ function $f : M \to N$ is also sometimes denoted by f_* and called the pushforward. The differential at each point $m \in M$ is a linear function on tangent spaces, $f_{*m} : T_m M \to T_{f(m)}N$. (Really it's a vector bundle homomorphism between the two tangent bundles.) We can then define it's dual map at each point m which we call the *pullback* $f_m^* : T_f(m)N^* \to T_m M^*$.

The importance of the pullback is that we can use it to "pull back" differential k-forms on N to differential k-forms on M. Suppose $\alpha \in \Omega^k(N)$, then we define its pullback by f to be the form $f^*\alpha \in \Omega^k(M)$ given by $f^*\alpha_m(X_1, \dots X_k) = \alpha_{f(m)}(f_*X_1, \dots, f_*X_k)$ for each $m \in M$ and $X_i \in T_m M$. Thought of this way, we can consider the pullback of a C^∞ function $g : M \to N$ as a mapping $g^* : \Omega^*(N) \to \Omega^*(M)$. Furthermore, it turns out to be an algebra homomorphism. i.e, for differential forms $\alpha, \beta \in \Omega^*(N)$ and $\mu, \lambda \in \mathbb{R}$ we have that $f^*(\mu\alpha + \lambda\beta) = \mu f^*(\alpha) + \lambda f^*(\beta)$ and $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$

An important fact of the exterior derivative is that for any smooth map $\psi : M \to N$, ψ^* commutes with d. That is, for any $\alpha \in \Omega^*(N)$ we have that $d(\psi^*(\alpha)) = \psi^*(d(w))$.

2.3 Singular Simplexes

The standard k-simplex Δ^k is defined to be

$$\Delta^k = \begin{cases} \{(x_1, \dots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k x_i \leq 1 \text{ and each } x_i \geq 0\} & : k > 0 \\ \{0\} & : k = 0 \end{cases}$$

For a differentiable manifold M, a differentiable (or smooth) singular k-simplex is a map $\sigma : \Delta^k \to M$ which extends to a C^∞-mapping on a neighbourhood of Δ^k. We will simply call this a k-simplex.

We can take the set of all formal linear combinations (with coefficients in $\mathbb{R}$) of k-simplexes which we will denote by $S_k(M;\mathbb{R})$. An element of this vector space is of the form $\sum_{i=1}^m a_i\sigma_i$ and is called a k-chain. The set of all chains is denoted by $S_*(M;\mathbb{R})$.

We can define a boundary operator ∂ on simplexes. Suppose we have some k-simplex σ. We define σ^i to be the its ith face. i.e, we restrict σ onto the ith face of the standard k-simplex. Then we define the boundary operator as $\partial\sigma = \sum(-1)^i\sigma^i$ giving us a $(k-1)$-simplex, the alternating sign keeps track of orientation. This can then be extended linearly to chains.

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The boundary operator has the important property that $\partial \circ \partial = 0$. This is because of the way it is defined as an alternating sum. Applying it twice causes all the parts to cancel out leaving 0.

The importance of simplexes is that we can use them integrate differential forms. Before we define the general case, let's first look at the simple case of a differential n-form defined on an open subset $U \subset \mathbb{R}^n$. Because in local coordinates, an n-form on n-dimensional space is of the form $f(x_1, \ldots, x_n) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ where f is $C^\infty(U)$. we can simply define the integral of this n-form as the Riemann nth fold integral of f over U. We can then use simplexes to extend this to general manifolds.

Definition 2.4. Let M be a smooth manifold. Suppose σ is an n-simplex on M and ω a differential n-form on M. Define the integral of ω over σ as:

$$\text{When } n = 0 \quad \int_\sigma \omega = \omega(\sigma(0))$$

$$\text{When } n \geq 1 \quad \int_\sigma \omega = \int_{\Delta^n} \sigma^*(\omega)$$

We then extend this definition to chains by linearity. i.e, if $c = \sum a_i \sigma_i$ is a chain then $\int_c \omega = \sum a_i \int_{\sigma_i} \omega$

Remark. In the above definition, because σ itself is not generally smooth, the pullback σ^* is not always defined. We instead mean that σ^* is the pullback of the smooth function that σ extends to.

2.4 Sequences, (Co)Chains and (Co)Homology

Let R be any commutative ring. Suppose we have a collection $\{A_i | i \in \mathbb{Z}\}$ of $\mathcal{R}$-modules along with homomorphisms $A_{n-1} \to A_n$. This gives us a sequence of modules which we write diagrammatically as

$$\ldots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \ldots$$

We say a sequence is *exact* at A if for $f : B \to A$ and $g : A \to C$, then $im(f) = ker(g)$. An exact sequence is one which is exact at all its component sets.

A *short exact sequence* is an exact sequence of $\mathcal{R}$-modules of the form:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

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We now define important types of sequences. that of chain and cochain complexes.

Definition 2.5. A *Chain Complex*, denoted by (C_*, d_*), is a sequence of $\mathcal{R}$-modules $(\dots, C_{n-1}, C_n, C_{n+1}, \dots)$ with decreasing index n called its degree, connected by homomorphisms $d_n : C_n \to C_{n-1}$ called the *nth boundary operators* such that $d_{n-1} \circ d_n = 0$ for all n.

$$\dots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

A *Cochain* complex (C^*, d^*) is defined similarly except the degree n increases instead of decreases, the homomorphisms are called *coboundary operators* and by convention the index is written as a superscript instead of a subscript. e.g,

$$\dots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \dots$$

The reason for this convention is that if you take a chain complex (A_*, d_*). You can form a cochain complex by taking the dual module for each A_n and the dual function for each d_n. Therefore Cochains are a sort of dual to chains.

Given some chain complex $(A_\star, d_\star)$, if we have some element $a \in A_q$ such that $d(a) = 0$, then a is called a *qth degree cycles* while if there exists an element $b \in A_{q+1}$ such that $d(b) = a$, then a is called a *qth degree boundary*. For cochains we define cocycles and coboundaries in exactly the same way.

Notice that because of the property $d \circ d = 0$, boundaries are cycles and coboundaries are cocycles. So we can form the quotient space $ker(d_n)/im(d_{n-1})$ which we denote by $H_n(C_*)$ which we call the *nth homology module* of C_*. We similarly define the *nth cohomology module* $H^n(C^*)$ of a cochain complex C^*.

Remark. The homology modules $H_n(C_*)$ and cohomology modules $H^n(C^*)$ give us a measure of how much the sequence fails to be exact at C_n and C^n since if it is exact, then each (co)cycle would also be a (co)boundary and these modules would then be trivial.

Complexes themselves are objects and so we can define mappings between complexes in a meaningful way.

Definition 2.6. A map between two chain complexes C_* and D_* is called a *Chain Map* and is a collection of homomorphisms $\{f_n : C_n \to D_n\}$ such that the resulting diagram commutes

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$$\begin{array}{ccccccccc} \ldots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \ldots \end{array}$$

A *Cochain map* is defined similarly.

Notice that that a cochain map $f : C^* \to D^*$ sends q-cocycles of C^* to q-cocycles of D^* due to commutativity of the diagram. Similarly, f sends coboundaries to coboundaries. So a cochain map induces a well defined mappings between the cohomology modules which we tend to denote by $f_q^* : H^q(C^*) \to H^q(D^*)$.

There are two important lemmas that we will need. Proofs of these are abundant in the literature but very readable ones can be found at [4].

Lemma 2.1 (the Five Lemma). *Given a commutative diagram of modules in the form of*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

where each of the rows are exact and f_1, f_2, f_4 and f_5 are isomorphisms. Then f_3 is an isomorphism.

Lemma 2.2 (ZigZag Lemma for cochains). *A short exact sequence of Cochain maps $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ (That is, each $0 \longrightarrow A^q \longrightarrow B^q \longrightarrow C^q \longrightarrow 0$ is exact) gives rise to a long exact sequence in cohomology:*

$$\ldots \longrightarrow H^{n-1}(B^*) \longrightarrow H^{n-1}(C^*) \xrightarrow{\partial_{n-1}} H^n(A^*) \longrightarrow H^n(B^*) \longrightarrow \ldots$$

Where ∂_n are called the connecting homomorphisms. A similar lemma holds for chains as well.

The homomorphisms that are not the connecting homomorphisms in the long exact sequence of cohomology are just the induced maps of the cochain maps. It is not important to us to know exactly what the connecting homomorphisms are.(They are explained in [4]).

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3 De Rham's Theorem

3.1 Singular and De Rham Cohomology (Mayer vietoras sequence)

3.1.1 Cohomology

Notice that for a smooth manifold M, $\Omega^*(M)$ along with the exterior derivative d forms a cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots$$

So we can form cohomology modules from this complex which we call *De Rham* Cohomology groups (Actually vector spaces but they are called groups for historical reasons). We denote the qth De Rham cohomology group by $H^q_{DR}(M)$. We also call cocycles *closed* and coboundaries *exact*.

Similarly, we get a chain complex of $S_*(M;\mathbb{R})$ along with the boundary operator ∂.

$$\dots \xrightarrow{\partial} S_2(M;\mathbb{R}) \xrightarrow{\partial} S_1(M;\mathbb{R}) \xrightarrow{\partial} S_0(M;\mathbb{R}) \longrightarrow 0$$

Where the corresponding qth homology module $H_q(M;\mathbb{R})$ are called the *qth singular homology group* (similarly a vector space).

However, it turns out we are more interested in the corresponding cochain complex. Let $S^k(M;\mathbb{R}) = \hom(S_k(M;\mathbb{R}), \mathbb{R})$ and ∂^* the dual map of the boundary operator ∂. Then we get the cochain complex

$$0 \longrightarrow S^0(M;\mathbb{R}) \xrightarrow{\partial^*} S^1(M;\mathbb{R}) \xrightarrow{\partial^*} S_3(M;\mathbb{R}) \xrightarrow{\partial^*} \dots$$

This gives us the *qth singular cohomology groups* $H^q(M;\mathbb{R})$.

3.1.2 Mappings between cohomology and homotopy

Now consider a smooth map $f : M \to N$ between two smooth manifolds M and N. From above we know that f^* commutes with the exterior derivative d. If we think of f^* as a collection of functions $\{f_k^* : \Omega^k(N) \to \Omega^k(M)\}$ then f^* can be thought of as a cochain mapping from $\Omega^*(N)$ into $\Omega^*(M)$.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \Omega^{k-1}(N) & \longrightarrow & \Omega^k(N) & \longrightarrow & \Omega^{k+1}(N) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f^*_{k-1}} & & \downarrow{\scriptstyle f^*_k} & & \downarrow{\scriptstyle f^*_{k+1}} & & \\
\dots & \longrightarrow & \Omega^{k-1}(M) & \longrightarrow & \Omega^k(M) & \longrightarrow & \Omega^{k+1}(M) & \longrightarrow & \dots
\end{array}$$

So this in turn induces mappings $f^*_k : H^k_{DR}(N) \to H^k_{DR}(M)$ between the de rham cohomologies. We use the same notation for both the pullback and the induced maps between cohomologies. Which one we mean is easily understood through context.

The smooth map f also induces mappings between the singular homologies and cohomologies. Consider the functions $\tilde{f}_k : S_k(M;\mathbb{R}) \to S_k(N;\mathbb{R})$ given by the mapping $\tilde{f}_k(\sigma) = f \circ \sigma$ for k-simplex σ which we extend linearly. Now each $\tilde{f}_k$ is a linear map and because

$$\tilde{f}_k \circ \partial(\sigma) = \tilde{f}_k(\sum_i (-1)^i \sigma^i) = \sum_i (-1^i)\tilde{f}_k(\sigma^i) = \sum_i (-1^i)(\tilde{f}_k(\sigma))^i = \partial \circ \tilde{f}_k(\sigma)$$

these then form a a chain map from $S_*(M;\mathbb{R})$ into $S_*(N;\mathbb{R})$ and so induce maps $f_{*k} : H_k(M;\mathbb{R}) \to H_k(N;\mathbb{R})$.

In a very similar way f also induces mappings between the singular cohomologies. The duals of $\tilde{f}_k$ are linear maps $\tilde{f}^*_k : S^k(N;\mathbb{R}) \to S^k(M;\mathbb{R})$ and these also commutes with the dual of the boundary operator. So f induces mappings $f^*_k : H^k(N;\mathbb{R}) \to H^k(M;\mathbb{R})$ between singular cohomologies.

An important property of both the singular and de rham cohomology groups is that they are homotopic invariants (See [2, Thm. 11.6, Thm. 11.27(c)]).

Here we compute the cohomology for the simplest of all smooth manifolds. The one point space $\{x\}$.

Theorem 3.1. *$H^q_{DR}(\{x\})$ and $H^q_{DR}(\{x\})$ are isomorphic to $\mathbb{R}$ when $q = 0$ and vanish when $q \neq 0$*

Proof. First let's compute the de rham cohomology groups. The one point space is a 0-dimensional smooth manifold and all maps are smooth on it. Since it has 0 dimensions $\Omega^k(\{x\}) = 0$ for $k > 0$ and so the only differentials are the set of all functions of

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the point x into $\mathbb{R}$. Therefore $H^0_{DR}(\{x\}) \cong \mathbb{R}$.

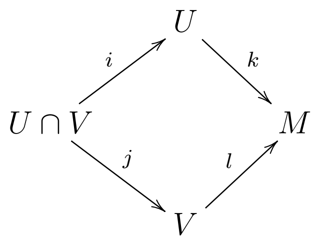
To compute the singular cohomology groups, we note that there is only one k-simplex into the space $\{x\}$ which is the constant function. So the set of k-chains is then isomorphic to $\mathbb{R}$. i.e, the set of all scalar multiples of this simplex. This then implies that $S^k(\{x\}; \mathbb{R}) \cong \mathbb{R}$ for all $k \geq 0$. Now, consider a k-simplex σ where $k > 0$. Its boundary is $\partial\sigma = \sum_{i=0}^{k}(-1)^i \sigma^i$ but each σ^i must also be the same since there is only one possible map they can be. Therefore ∂ on odd k-chains is the zero map, while on even k-chains it's an isomorphism. This then forces $H_0(\{x\}; \mathbb{R}) \cong \mathbb{R}$ since all 0-chains are 0-cycles but none except 0 are 0-boundaries. When k is odd, $H_k(\{x\}; \mathbb{R}) = 0$ since all ∂ being the zero map makes all k-chains cycles and because the boundary with a higher index is an isomorphism, all these cycles are boundaries. Similarly, when k is even, only 0 is a cycle and this is obviously also a boundary.

We simply take duals to prove this for singular cohomology. □

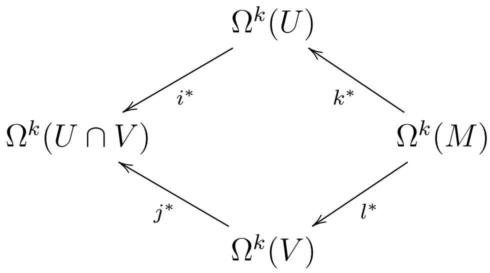
The importance of the above theorem is to realise that open convex sets in $\mathbb{R}^n$ are homotopy equivalent to the one point space $\{x\}$.

3.1.3 Mayer-Vietoris Sequences

Now consider the following the following diagram where each i,j,k and l are the obvious inclusion mappings.



These pull back to restriction mappings i^*, j^*, k^* and l^* on the differential k-forms.



Which gives us the sequence for all k:

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0$$

it is routine to prove that that this is exact for all k and because the exterior derivative commutes with pullbacks, we get a short exact sequence of cochain maps. By the zigzag lemma we get a long exact sequence in cohomology called the Mayer-Vietoris sequence for de rham cohomology.

$$\ldots \longrightarrow H^{n-1}_{DR}(U) \oplus H^{n-1}_{DR}(V) \longrightarrow H^{n-1}_{DR}(U \cap V) \longrightarrow H^n_{DR}(M) \longrightarrow H^n_{DR}(U) \oplus H^n_{DR}(V)) \longrightarrow \ldots$$

Using the same ideas as above, we can also get a Mayer-Vietoris sequence for singular cohomology as well.

3.2 Stokes Theorem and the De Rham homomorphism

One of the most famous theorems is Stokes theorem. As we will see, it gives us a very natural mapping between the singular and de rham cohomologies.

Theorem 3.2 (Stokes Theorem). *Let σ be some k-chain in a smooth manifold M, and ω a smooth $(k-1)$ form defined on a neighbourhood of $c(\Delta^k)$. Then,*

$$\int_{\partial c} \omega = \int_c d\omega$$

For a smooth manifold M, we can define homomorphisms $l_k : \Omega^k(M) \to S^k(M; \mathbb{R})$ for each $k \in \mathbb{Z}$ by mapping $\omega \mapsto \int \omega$. i.e, for any k-form ω, we treat $\int \omega : S_q(M; \mathbb{R}) \to \mathbb{R}$ as a linear function on singular k-chains.

Stokes theorem then implies that this collection of homomorphisms $\{l_k\}$ commutes with the exterior derivative and boundary operators, and so it is a cochain map.

$$
\begin{array}{ccccccccc}
\ldots & \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \xrightarrow{d} & \ldots \\
 & & \downarrow{\scriptstyle l_{k-1}} & & \downarrow{\scriptstyle l_k} & & \downarrow{\scriptstyle l_{k+1}} & & \\
\ldots & \xrightarrow{\partial^*} & S^{k-1}(M;\mathbb{R}) & \xrightarrow{\partial^*} & S^k(M;\mathbb{R}) & \longrightarrow & S^{k+1}(M;\mathbb{R}) & \xrightarrow{\partial^*} & \ldots
\end{array}
$$

Therefore, this induces mappings between the de rham and singular cohomology groups.

Definition 3.1. Let U be some smooth manifold. We denote the induced homomorphisms of the kth cohomology groups by $DR_k(U) : H^k_{DR}(U) \to H^k(U)$. We call the collection $\{DR_k(U)\}$ the De Rham homomorphism on U or simply $DR(U)$ and if each $DR_k(U)$ is an isomorphism then we say $DR(U)$ is an isomorphism.

In order to make use of the Mayer-Vietoris sequences we need the following two lemmas.

Lemma 3.1. *Let $\psi : M \to N$ be a C^∞ function between smooth manifolds M and N. Then this induces two pullbacks on the de rham and singular cohomology groups: $\psi^*_k : H^k_{DR}(N) \to H^k_{DR}(M)$ and $\psi^*_k : H^k(N;\mathbb{R}) \to H^k(M;\mathbb{R})$ which commute with the kth De rham homomorphism on N and M. e.g, the following commutes:*

$$
\begin{array}{ccc}
H^k_{DR}(N) & \xrightarrow{\psi^*_k} & H^k_{DR}(M) \\
\downarrow{\scriptstyle DR_k(N)} & & \downarrow{\scriptstyle DR_k(M)} \\
H^k(N;\mathbb{R}) & \xrightarrow{\psi^*_k} & H^k(M;\mathbb{R})
\end{array}
$$

Proof. This is a consequence of the calculation for k-form ω on N and k-simplex σ in M.

$$\int_\sigma \psi^*\omega = \int_{\Delta^k} \sigma^*\psi^*\omega = \int_{\Delta^k} (\psi\sigma)^*\omega = \int_{\psi\sigma} \omega$$

□

An important consequence of this lemma is that if M and N are diffeomorphic and ψ a diffeomorphism, then both horizontal ψ^* are isomorphic. We can then conclude by commutativity that if $DR(M)$ is an isomorphism, then DR is an isomorphism on all smooth manifolds N diffeomorphic to M.

Lemma 3.2. *For a smooth manifold M and two open sets U and V whose union is M. We have that the following commutes for all k.*

$$\begin{array}{ccc} H^{k-1}_{DR}(U \cap V) & \longrightarrow & H^k_{DR}(M) \\ \downarrow & & \downarrow \\ H^{k-1}(U \cap V; \mathbb{R}) & \longrightarrow & H^k(M; \mathbb{R}) \end{array}$$

where the horizontal homomorphisms are the connecting homomorphisms from their respective Mayer-Vietoris sequence.

We omit the proof here since we haven't explicitly defined what the connecting homomorphisms are. (See [2, lem. 11.33])

3.3 Proof of De Rham's Theorem

The proof we give follows the same idea as the one in [2] but with modification. First we will prove 3 important Lemmas, before we finally prove De Rham's theorem.

Lemma 3.3. *If $U \subset \mathbb{R}^n$ and U is convex, then $DR(U)$ is an isomorphism.*

Proof. Because de rham and singular cohomology are homotopic invariants and U is homotopy equivalent to $\{x\}$, from Theorem 3.1 we have that the qth de rham and qth singular cohomology groups vanish for $q \neq 0$ and are isomorphic to $\mathbb{R}$ when $q = 0$.
Now consider the case when $q = 0$, in this case $H^0_{DR}(U)$ is the one-dimensional vector space of constant functions on U since the only 0-forms ω such that $d\omega = 0$ are the constant functions. Similarly, the only 0-simplexes are the maps from $\{0\} \to U$. Since by definition the integral of a 0-from over a 0-simplex is just the value of the form at that point the simplex maps 0 into. We find that the de rham homomorphism can't be the zero map and hence must be an isomorphism. □

Lemma 3.4. *Let M be any smooth manifold. Given a basis $\mathcal{B}$ on M, there exists a countable open cover $\{U_i\}$ of M such that each U_i can be written as the finite union of basis elements and if $U_i \cap U_j = \emptyset$ then $i \neq j \pm 1$*

Proof. Let $\{V_i\}$ be an open cover of M as in Theorem 2.1 and let ψ_i be a partition of unity subordinate to this cover. Let us define a new C^∞-function called α on M by setting

$$\alpha = \sum_{i=1}^{\infty} i\psi_i$$

Now, suppose $p \in M$ but $p \notin \cup_{i=1}^{N} supp(\psi_i)$ (which is compact by lemma) Then

$$\alpha(p) = \sum_{i=1}^{\infty} i\psi_i(p) = \sum_{i=N+1}^{\infty} i\psi_i(p) > \sum_{i=N+1}^{\infty} N\psi_i(p) \geq N \sum_{i=1}^{\infty} \psi_i(p) = N$$

Therefore $\alpha^{-1}([0, N])$ is compact for each $N \in \mathbb{N}$. Because α is C^∞, it is continuous and so $\alpha^{-1}(a, b)$ is open and must have compact closure for any open interval (a, b).

Define the sets $A_i = \alpha^{-1}(i+1/4, i+7/4)$ and $A_i' = \alpha^{-1}(i, i+2)$ for $i = -1, 0, 1, 2, ...$ Now, for each point $x \in \overline{A_i}$, take a basis element $B_x \in \mathcal{B}$ that contains x but is contained in A_i' and form the open cover $\{B_x\}$ of $\overline{A_i}$. Since $\overline{A_i}$ is compact, there is a finite subcollection of $\{B_x\}$ that still covers $\overline{A_i}$. Take U_i as the union of this finite subcollection and because we chose each B_x to be contained within A_i' we find that $A_i \subseteq U_i \subseteq A_i'$. Therefore, if $U_i \cap U_j = \emptyset$ then $i \neq j \pm 1$. □

Lemma 3.5. *Let M be a smooth manifold. Suppose $M = \bigcup_{i=1}^{k} U_i$ where $k \in \mathbb{N}$ and U_i are open. If DR is an isomorphism on each of the sets $\{U_i\}$ and each finite intersection of these sets, then $D(M)$ is an isomorphism.*

Proof. It is sufficient to simply prove this true for the case $k = 2$ since the general case can then be proven by induction.

Suppose $M = U \cup V$ where U, V are open and DR is an isomorphism on U, V and $U \cap V$. From lemma 3.1 and 3.2 we get this commutative diagram between the two Mayer-Vietoris sequences of de rham and singular cohomology.

$$\begin{array}{ccccccccc}
H_{DR}^{q-1}(U) \oplus H_{DR}^{q-1}(V) & \longrightarrow & H_{DR}^{q-1}(U \cap V) & \longrightarrow & H_{DR}^{q}(M) & \longrightarrow & H_{DR}^{q}(U) \oplus H_{DR}^{q}(V) & \longrightarrow & H_{DR}^{q}(U \cap V) \\
\downarrow{\scriptstyle DR_{q-1}(U)\oplus DR_{q-1}(V)} & & \downarrow{\scriptstyle DR_{q-1}(U\cap V)} & & \downarrow{\scriptstyle DR_q(M)} & & \downarrow{\scriptstyle DR_q(U)\oplus DR_q(V)} & & \downarrow{\scriptstyle DR_q(U\cap V} \\
H^{q-1}(U) \oplus H^{q-1}(V) & \longrightarrow & H^{q-1}(U \cap V) & \longrightarrow & H^{q}(M) & \longrightarrow & H^{q}(U) \oplus H^{q}(V) & \longrightarrow & H^{q}(U \cap V)
\end{array}$$

By hypothesis, we know that all the vertical homomorphisms are isomorphims except $DR_q(M)$. But the rows are exact and so the Five Lemma implies that $DR_q(M)$ is isomorphic. □

Theorem 3.3 (De Rham's Theorem)**.** *Let M be any smooth manifold. Then $DR(M)$ is an isomorphism.*

Proof. First we show that if $DR(U)$ is an isomorphism for a each U in some countable collection, it is an isomorphism for the disjoint union.

Let $\{M_j\}$ be a countable collection of manifolds where $DR(M_j)$ is an isomorphism for each j. Let $M = \coprod_j M_j$ be the disjoint union of these manifolds. Denote the inclusion maps by $i_j : M_j \to M$. Then the map $i = (i_1, i_2, \dots)$ induces isomorphisms between $\oplus_j H^k_{DR}(M_j)$ and $H^k_{DR}(M)$ as well as $\oplus_j H^k(M_j; \mathbb{R})$ and $H^k(M; \mathbb{R})$. For each k, $\oplus_j DR_k(M_j)$ is an isomorphism between the direct product of the de rham and singular cohomology groups and so by Lemma 3.1, $DR_k(M)$ must also be an isomorphism.

Now, given $\{U_i\}$ be an open cover as in lemma 3.4. let $U_{odd} = \cup U_{2k+1}$, $U_{even} = \cup U_{2k}$ and $U_{int} = \cup(U_k \cap U_{k+1})$ for $k \in \mathbb{N}$. Notice that $U_{odd} \cap U_{even} = U_{int}$. So by lemma 3.5, if $DR(U_{even}$, $DR(U_{odd})$ and $DR(U_{int})$ were isomorphic then so would $DR(M)$. Therefore we only need to show DR is isomorphic on each U_k and $U_k \cap U_{k+1}$ since U_{odd}, U_{even} and U_{int} are disjoint unions of these sets.

It is sufficient to simply show that M has a basis with the property that DR is isomorphic on each basis element and on each finite intersection of basis elements. This is because each U_k can be written as the union of finitely many basis elements and $U_k \cap U_{k+1}$ can be written as the union of finitely many intersections of these basis elements. Which by lemma 3.5 would imply that DR is an isomorphism on these sets.

If M is an open subset of $\mathbb{R}^n$ for some integer n. Then M does have a basis with this property. Simply note that M would then have a basis of n-balls and since the intersection of balls is still convex, by Lemma 3.3 DR is isomorphic on these intersections.

When M is a general smooth manifold of dimension n, we can simply take a basis of domain charts. Each of these domains (and finite intersections) are diffeomorphic to an open subset of $\mathbb{R}^n$ and hence DR is isomorphic on this basis. Hence the theorem is proven. □

References

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