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The Curve Shortening Flow

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I undertook a project over summer at the Australian National University under the supervision of Dr. Ben Andrews. The aim was to gain an understanding of Grayson's theorem regarding the curve shortening flow.

First I took a crash course in differential geometry as applied to curves, concentrating in particular on plane curves. There are a number of interesting classic theorem's in this area. Of particular note for my project is the so called "Isoperimetric Inequality". This states that given all closed curves of length L, which bounds the greatest area. The answer is the circle. This was known to the ancient Greek's, but was not proven satisfactorily until Weistrass, in 1870 used the Calculus of Variations to show that a solution existed. Since then, a number of proofs have been given with varying assumptions (smoothness, convextity etc.).

Next I looked at the curve shortening flow. For this we take a curve and evolve it in time with velocity at each point equal to the curvature along the inward normal direction. In particular, I looked at a paper by Michael Gage from 1983 entitled "An isoperimetric inequality with applications to curve shortening". In this paper, Gage first proves an inequality relating the integral over the curvature squared, the curve length and enclosed area. This is then applied to solutions of the curve shortening flow to show that the square of the curve length divided by the area decreases. This in effect, by the "classic" isoperimetric inequality discussed in the previous paragraph results in the curve becoming "rounder" in some sense.

With these preliminaries under my belt, I then examined a paper by Gerhard Huisken from 1998 entitled "A Distance Comparison Principle for Evolving Curves". In this paper, the Euclidean distance (called extrinsic distance in the paper) between two points is compared to the distance along the curve (intrinsic distance). It is shown that a local minimum of the extrinsic distance divided by the intrinsic distance is non-decreasing in time. This only applies for non-closed curves since the intrinsic distance is not smoothly defined for closed curves. However, a suitable "replacement" is chosen which is smoothly defined for closed curves and again the ratio is non-decreasing.

This amounts (by a maximum principle for parabolic equations) to proving that the infinum of this ratio is non-decreasing. This result is then used to show that only a certain kind of singularity may occur under the flow and that in this situation it has been proven that the curve approaches a circle before collapsing to a point. This is Grayson's theorem.

All in all, the experience was very rewarding. The Big Day In was a lot of fun and, as I discovered showed that I'm not the only mathematics student who enjoys the odd beer or 12.

Paul received an ICE-EM Vacation Scholarship in December 2006.
See http://www.ice-em.org.au/students.html#scholarships2007