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Applications of Bass-Serre Theory to C^*-algebras

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April 24, 2013

1 Introduction

The theory of C^*-algebras was first developed in the 1940's through the study of operators arising in quantum mechanics. They have since become their own important area of research in the field of functional analysis. C^*-algebras do not have a classification theorem and so examples are important in trying to analyse and classify the structure of these abstract spaces. Following the lead of directed graph C^*-algebras, my project aimed to construct C^*-algebras from an object in geometric group theory known as a graph of groups. These are a key tool for determining the structure of infinite groups. Bass-Serre theory provides an equivalence between graphs of groups and quotients of a group actions on trees. We aimed to exploit the dynamics of this action in constructing a universal C^*-algebra associated to a graph of groups.

The overall goal of my project was to try and generate a universal C^*-algebra, $C^*(\mathcal{G})$, from a graph of groups $\mathcal{G}$ in such a way that an analogue of the following theorem about directed graph C^*-algebras holds.

Theorem 1.1. *Let E be a directed graph with universal covering tree T. Let ∂T be the boundary of T and $\pi_1(E)$ the fundamental group of E. Then,*

$$C^*(E) \sim_{sme} C_0(\partial T) \rtimes \pi_1(E)$$

where $\sim_{sme}$ is strong Morita equivalence.

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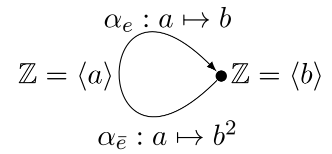
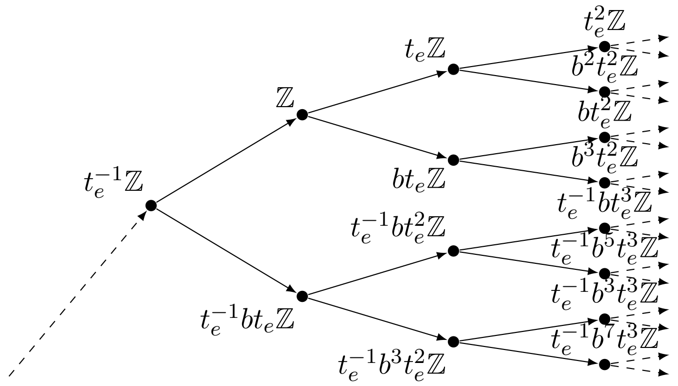
2 Our Example

In search of a general procedure to generate a universal C^*-algebra from a graph of groups, my project focused primarily around one particular graph of groups $\mathcal{G} = (\Gamma, G)$; a loop with $\mathbb{Z}$ on both the vertex v and edge e with the maps α_e and $\alpha_{\bar{e}}$ being the identity and $\mathbb{Z} \mapsto m\mathbb{Z}$ respectively. This can be visualised by:



The fundamental group of $\mathcal{G}$ is the Baumslag-Solitar group [3], $BS(1,m) := \langle t_e, b | t_e b = b^m t_e \rangle$ and the universal covering tree $T_{\mathcal{G}}$ associated to $\mathcal{G}$ is an infinite complete binary tree.



The boundary of the universal covering tree is given by:

$$\partial T_{\mathcal{G}} = \{g_1 x_1 g_2 x_2 \cdots : x_i \in \{e, \bar{e}\}, g_i \in T_{x_i}, r(x_{i+1}) = s(x_i), \\ \text{and } x_{i+1} = \overline{x_i} \Longrightarrow g_{i+1} \neq 0\}.$$

which is a topological space of tail equivalent paths, homeomorphic to the m-adic numbers $\mathbb{Q}_m$.We denote by $\partial T_{\mathcal{G}}^*$ the set of finite paths:

$$\partial T_{\mathcal{G}}^* = \{g_1 x_1 \ldots g_n x_n : x_i \in \{e, \bar{e}\}, g_i \in T_{x_i}, r(x_{i+1}) = s(x_i), \\ \text{and } x_{i+1} = \overline{x_i} \Longrightarrow g_{i+1} \neq 0\}.$$

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For each $\mu \in \partial T_{\mathcal{G}}^*$ we denote by $Z(\mu)$ the cylinder set $Z(\mu) = \{g_1 x_1 \cdots \in \partial T_{\mathcal{G}} : g_1 x_1 \ldots g_{|\mu|} x_{|\mu|} = \mu\}$.

There is a natural action $\gamma : BS(1,m) \to \mathrm{Aut}(T_{\mathcal{G}})$ given by Bass-Serre theory [2,5]. This extends to an action on $\partial T_{\mathcal{G}}$ and induces an action $\overline{\gamma} : BS(1,m) \to \mathrm{Aut}(C_0(\partial T_{\mathcal{G}}))$, where $C_0(\partial T_{\mathcal{G}})$ is the continuous functions vanishing at infinity on the locally compact space $\partial T_{\mathcal{G}}$, a commutative C^*-algebra. We can then form the C^*-dynamical system $(C_0(\partial T_{\mathcal{G}}), BS(1,m), \overline{\gamma})$ which enables us to construct the crossed product, $C_0(\partial T) \rtimes_{\overline{\gamma}} BS(1,m)$ [1].

We aim to use a Cuntz-Krieger construction to build our universal C^*-algebra, similarly to directed graph C^*-algebras. We assign e and $\bar{e}$ partial isometries s_e and $s_{\bar{e}}$, v the identity 1 and to b, the generating element b of $\mathbb{Z}$ we assign the unitary u_b. Then $C^*(\mathcal{G}) := \overline{\mathrm{span}}\{s_e, s_{\bar{e}}, 1, u_b\}$ subject to the relations,

$$\begin{aligned}
1 &= s_e^* s_e + s_{\bar{e}} s_{\bar{e}}^* \\
1 &= s_{\bar{e}}^* s_{\bar{e}} + s_e s_e^* \\
s_e^* s_e &= s_e s_e^* + \sum_{i=1}^{m} u^i s_{\bar{e}} (u^i s_{\bar{e}})^* \\
s_{\bar{e}}^* s_{\bar{e}} &= \sum_{i=0}^{m} u^i s_{\bar{e}} (u^i s_{\bar{e}})^* \\
u s_e &= s_e u^m
\end{aligned}$$

We proposed the following lemmas,

Lemma 2.1. *Let:*

$$\begin{aligned}
&s_e := i_A(\mathcal{X}_{Z(1e)}) i_H(t_e), \\
&s_{\overline{e}} := i_A(\mathcal{X}_{Z(1\overline{e})}) i_H(t_e)^*, \ \textit{and} \\
&u_b = i_H(b),
\end{aligned}$$

Where $\mathcal{X}_S$ is the indicator function on a the set S and (i_A, i_H) is the covariant homomorphism which generates $C_0(\partial T) \rtimes_{\overline{\gamma}} BS(1,m)$. Then s_e, $s_{\overline{e}}$ and u_b satisfy the previous relations.

For each $\mu = g_1 x_1 \ldots g_n x_n \in \partial T_{\mathcal{G}}^*$ we denote $s_\mu := u_{g_1} s_{x_1} \ldots u_{g_n} s_{x_n} \in C^*(\mathcal{G})$.

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Lemma 2.2. *There is a non degenerate representation* $\rho : C_0(\partial T_{\mathcal{G}}) \to C^*(\mathcal{G})$ *characterised by:*

$$\rho(\mathcal{X}_{Z(\mu)}) = s_\mu s_\mu^* \text{ for each } \mu \in \partial T_{\mathcal{G}},$$

and a unitary representation $v : BS(1, m) \to C^*(\mathcal{G})$ *satisfying:*

$$v_{t_e} = s_e + s_{\bar{e}}^* \text{ and } v_g = u_g \text{ for each } g \in G_v \cong \mathbb{Z}.$$

Moreover, (ρ, v) *is a covariant representation.*

These lemmas suggest some sort of equivalence between $C^*(\mathcal{G})$ and $C_0(\partial T_{\mathcal{G}}) \rtimes_{\overline{\gamma}} BS(1, m)$.

3 General case

In the future we plan on extending the results we obtained to a more general case. For a general graph of groups $\mathcal{G} = (\Gamma, G)$ we assign each edge $e \in \Gamma$ a partial isometry s_e, each vertex $v \in \Gamma$ we assign a projection p_v, and for each g in the vertex groups G_v we assign a partial unitary u_g. We believe the relations for this case should be,

$$\begin{aligned}
p_{s(e)} &= s_e^* s_e + s_{\bar{e}} s_{\bar{e}}^* \\
s_e^* s_e &= \sum_{\substack{r(f)=s(e) \\ g \in G_{s(e)}/\alpha_f(G_f) \\ (g,f) \neq (\mathrm{id}_{G_{s(e)}}, \bar{e})}} u_g s_f (u_g s_f)^* \\
u_{\alpha_e(g)} s_e &= s_e u_{\alpha_{\bar{e}}(g)} \quad \forall g \in G_e \\
u_g u_g^* = u_g^* u_g &= \sum_{r(\mu)=e} p_\mu
\end{aligned}$$

4 Acknowledgements

I would like to thank AMSI for the opportunity to undertake this summer vacation research scholarship. The scholarship has helped me to both experience and discover what a professional career in mathematical research entails. I would also like to thank my supervisors David Pask and Nathan Brownlowe for academic guidance through this program.

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