COHOMOLOGY OF THE PINWHEEL TILING

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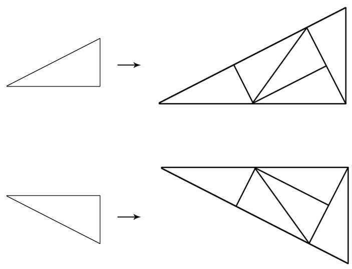
A tiling is a collection of subsets of $\mathbb{R}^d$, called tiles, for which any intersection of two tiles has empty interior and whose union is the d-dimensional Euclidean space. The most popular way to produce tilings is to use a substitution rule; that is, a rule for subdividing and expanding tiles which can be used to systematically tile Euclidean space. Under mild conditions, substitution tilings give rise to a dynamical system that is homeomorphic to the product of a Cantor set and a disc. Moreover, the substitution rule extends to a homeomorphism on the dynamical system itself. Two seminal examples of substitution tilings are the Penrose tiling and the pinwheel tiling. The Penrose tiling turns out to have 10 fold rotational symmetry while the pinwheel tiling has infinite rotational symmetry. Having tiles in an infinite number of distinct orientations has made the tiling cohomology of the pinwheel tiling difficult to compute.

The pinwheel tiling was discovered by John Conway and Charles Radin [3] and the following diagram shows the substitution rule.



The Anderson-Putnam complex [1] is used to compute the cohomology of tilings and has been generalised, for example in [2], to deal with tilings with infinite rotational symmetry, like the pinwheel tiling. For the pinwheel tiling we study two topological spaces Ω and Ω_0. Ω consists of all euclidean motions of the pinwheel tiling and Ω_0 is Ω modulo rotation; that is, two tilings in Ω are identified in Ω_0 if they are a rotation of each other. In this project we calculate the cohomology of Ω_0 by describing the substitution matrices and the boundary maps of the associated Anderson-Putnam complex. In order to do this we need a relabelling of the pinwheel tiling which forces its border, a complicated property required for cohomology calculations. In an unpublished manuscript, D. Frettlöh and M. Whittaker have described a relabelling of the pinwheel tiling which does force its border and we use this relabelling throughout the project.

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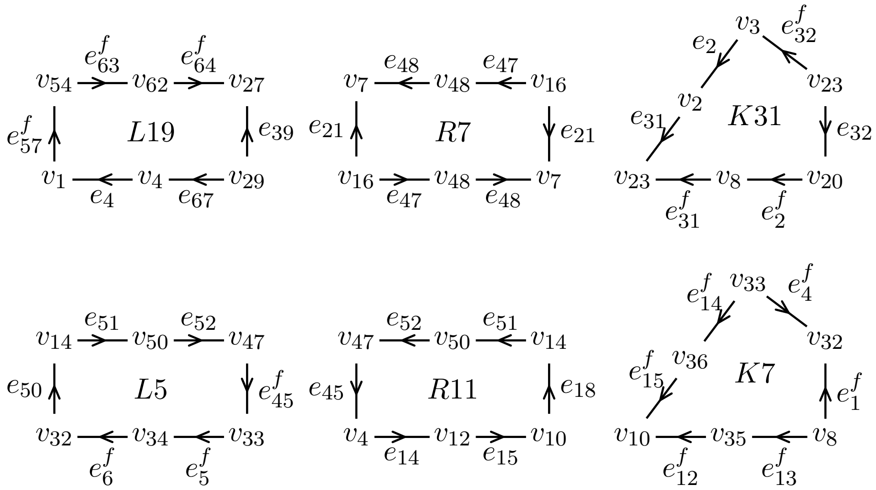
The new relabelled tiles are either the kites or rectangles which result from the removing the hypotenuses from the triangles appearing in the pinwheel tiling. The relabelling consists of 83 tiles, 138 edges and 73 vertices and the diagram below shows a sample of six tiles.



Using these tiles we can calculate the boundary and substitution matrices. The Anderson-Putnam complex requires us to compute the cohomology of Γ, a topological space which can be built from vectices, edges and faces (tiles). Let C^0, C^1 and C^2 denote the group of functions from the vertices, edges and tiles respectively into the integers. Let $\delta_1 : C^0 \to C^1$ and $\delta_2 : C^1 \to C^2$ be the boundary maps and we have the following cell complex,

$$C^0 \xrightarrow{\delta_1} C^1 \xrightarrow{\delta_2} C^2$$

This enables us to compute the cohomology groups of Γ, denoted $H^i(\Gamma)$. Now let A_0, A_1 and A_2 denote the substitution maps for the vertices, edges and tiles respectively and from this we can compute the cohomology group $H^i(\Omega_0)$ via the relation,

$$\check{H}^i(\Omega_0) = \lim_{\rightarrow}(H^i(\Gamma), A_i^*)$$

where $A_i^* : H^i(\Gamma) \to H^i(\Gamma)$ is the induced map on the the cohomology groups from the corresponding substitution.

Now we list our results,

$$H^0(\Gamma) \cong \mathbb{Z}$$
$$H^1(\Gamma) \cong \mathbb{Z}$$
$$H^2(\Gamma) \cong \mathbb{Z}^{18} \oplus \mathbb{Z}_2$$
$$\check{H}^0(\Omega_0) \cong \mathbb{Z}$$
$$\check{H}^1(\Omega_0) \cong \mathbb{Z}$$
$$\check{H}^2(\Omega_0) \cong \mathbb{Z}[\frac{1}{25}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_2$$

We note that the cohomology of the pinwheel tiling has been computed using spectral sequences in [2]. Using their results we can calculate the cohomology of Ω.

$$\check{H}^0(\Omega) \cong \mathbb{Z}$$
$$\check{H}^1(\Omega) \cong \mathbb{Z}^2$$
$$\check{H}^2(\Omega) \cong \mathbb{Z}[\frac{1}{25}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^6 \oplus \mathbb{Z}_2^5$$
$$\check{H}^3(\Omega) \cong \mathbb{Z}[\frac{1}{25}] \oplus \mathbb{Z}[\frac{1}{3}]^2 \oplus \mathbb{Z}^5 \oplus \mathbb{Z}_2$$

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