Non-linear Dynamics in a two-element Bistable Ring System

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Acknowledgements

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Taking part in the AMSI Vacation Research Scholarship has really opened my eyes to the vast possibilities in the study of mathematics. I have never been around a group of more enthusiastic students than during the incredible two days spent in Sydney for the CSIRO Big Day In. Listening to the presentations of my fellow AMSI Vacation Scholars has been tremendously inspiring and I would recommend students interested in the study of mathematics or statistics to apply for a position.

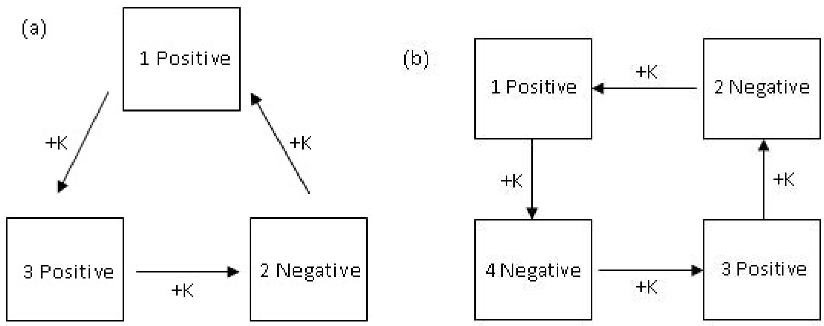

Figure 1: Ring system arrangements for (a) three elements with a frustration between elements 1 and 3, and (b) four elements with no frustrations.

1 Introduction

1.1 Background

Uni–directionally coupled bistable ring systems have been shown to undergo bifurcations resulting in oscillations when the control parameter, in this case the coupling strength K between the elements, is raised to a sufficiently high value. This non–linear response has triggered investigations into the possible use of these simple systems as affordable signal detectors due to their high sensitivity to external signals [4].

This has been studied in detail with ring systems of three bistable elements and in most of these ring systems that have been investigated, the coupling between elements was positive. Under these conditions, each element prefers to be in the opposing state to the element it is coupled with. Calling these states positive and negative, it can be seen that this preference can be satisfied when there are an even number of elements in the ring, for example with a four element system, they can be arranged as in Figure 1(b) and hence the system is stable. However, with an odd number of elements, a 'frustration' exists where two consecutive elements are in the same state, see Figure 1(a). These systems with frustrations can then begin to oscillate as the coupling strength is increased.

If instead, there is a mix of negatively coupled and positively coupled elements in the ring, those with negative coupling prefer to be in the same state whilst those with positive coupling prefer to be in the opposite state. As two different frustration types now exist, a frustration can occur in a ring system with an even number of elements. I will be investigating the dynamics of the most basic example of this group of systems, focusing on the ring system with two elements coupled together as in Figure 2. Here, element 1 prefers to be in the opposite state to element 2 whilst element 2 prefers the same state. Hence a frustration occurs and the system can exhibit auto-oscillations.

As the elements themselves are no different to those studied in previous research, I will model their bistability using the same quartic potential (1) and differential equation

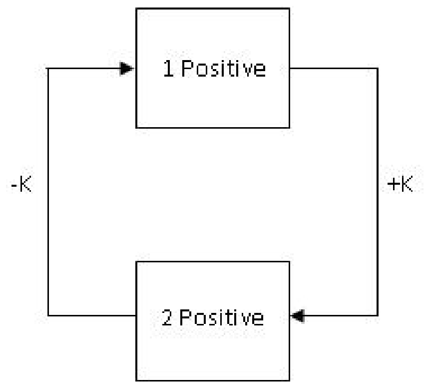

Figure 2: Two element frustrated ring system.

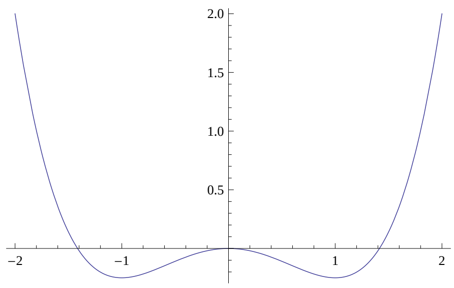

Figure 3: The quartic potential.

(2) as in [2]. As seen in Figure 3, the potential is symmetrical with each well being of the same depth.

$$U(x_i) = -\frac{1}{2}(x_i)^2 + \frac{1}{4}(x_i)^4 \tag{1}$$

$$\frac{dx_i}{dt} = \dot{x_i} = f(x_i) = -\nabla U \tag{2}$$

However, as the coupling is different (negative) for element 2, the coupling term in the differential equation's modelling the system must be modified. For positive coupling, the term $K(x_1 - x_2)$ concerns the difference between the elements' states, so the negative coupling will be modelled by replacing the difference with the sum $K(x_1 + x_2)$. This gives the differential equations:

$$\begin{aligned} \dot{x_1} &= x_1 - (x_1)^3 + K(x_1 - x_2) \\ \dot{x_2} &= x_2 - (x_2)^3 + K(x_1 + x_2) \end{aligned} \tag{3}$$

1.2 Areas of Focus

Before the response of this two element system to external signals can be assessed, the dynamics of the unforced differential equation system (3) must first be investigated, for comparison. In doing so, the fixed points of the system will be located and studied, and the bifurcation taking place, the critical coupling strength K_c and the relationship between the frequency of the oscillations and the coupling strength will be determined.

Following this, a small positive constant signal will be added, in two configurations, to see how the system's behaviour is affected, focusing on the same aspects of fixed points, bifurcations and oscillation frequency. Comparisons will then be made between the different systems and their behaviour.

In practice, noise will inevitably be present and as a result, the system must also be analysed in response to random noise. The noise will be added to the model in one configuration; matching, and numerical simulations will be carried out to view the behavioural changes.

2 Dynamics of the unforced system

In order to get a general idea of the behaviour of the system, numerical simulations have first been carried out with the use of MATLAB and Mathematica packages. Following this, the system will be analytically examined to locate the fixed points, analyse how they move with K and determine the bifurcation point.

2.1 Numerical simulation and fixed point analysis

To understand how the elements make their changes from each state once the oscillations have begun, the critical K at which the auto-oscillations begin to take place must be approximately located. To do this, I have chosen to analyse the fixed points and their stabilities.

A fixed point of a system is the point $\boldsymbol{x}^*$ at which all the $\dot{\boldsymbol{x}}$ in the system are equal to zero [5]. Applying the linearisation or Hartman-Grobman theorem, linearising the vector field $\dot{\boldsymbol{x}}$ about the fixed point by evaluating the Jacobian at that point and determining its eigenvalues and eigenvectors allows us to determine its stability [5]. When the eigenvalues have positive real part, the fixed point is unstable along the corresponding eigenvectors, and when they have negative real part, the fixed point is stable along the eigenvectors [5].

For the two-element ring system (3), out of a total of 9 fixed points, there is only one easily expressible solution at (0,0). The other 8 are extremely complicated and will not be explicitly defined. The Jacobian for this system is

$$\mathrm{D}f = \begin{pmatrix} 1+K-3\,(x_1)^2 & -K \\ K & 1+K-3\,(x_2)^2 \end{pmatrix}, \tag{4}$$

Evaluating at (0,0) gives

$$\mathrm{D}f(0,0) = \begin{pmatrix} 1+K & -K \\ K & 1+K \end{pmatrix}, \tag{5}$$

leading to eigenvalues $\lambda \in \{(1+K)-iK, (1+K)+iK\}$. As $K > 0$, these both have positive real part, hence the (0,0) fixed point is unstable and the two corresponding eigenvectors span an unstable 2D manifold.

Returning to the remaining fixed points, these can be found by treating the right hand sides of the differential equations (3) as simultaneous equations and equating each to zero (6). Solutions to x_1 and x_2 are then given by the use of Mathematica's solve facility.

$$\begin{aligned} x_1-(x_1)^3+K\,(x_1-x_2) &= 0 \\ x_2-(x_2)^3+K\,(x_1+x_2) &= 0 \end{aligned} \tag{6}$$

Using Mathematica's manipulate command, I was able to plot these solutions and view how they moved as K was varied. This is shown in Figure 4, for various values of

K, and appears to suggest a critical coupling strength of $K_c \simeq 0.547$. From the graphs, the fixed points appear to collide in pairs and disappear. This implies that saddle-node bifurcations have occurred [1].

Using this estimate of the critical coupling strength, MATLAB can be used to perform a numerical simulation of the behaviour of the elements shortly after the oscillations have begun. This results in the graph, Figure 5, which shows how element 1 switches states to try stay in the opposite state to element 2 whilst element 2 switches states to try stay in the same state as element 1. Performing the same simulation but with a subcritical value of K results in Figure 6 in which each element stays in a stable state. This stable state will depend on the initial conditions and in Figure 6, element 1 began in the positive state and element 2 the negative state.

Also included in Figures 5 and 6 is the sum of both signals for comparison with previous research; in this case, the summed response has the same period as each element whereas in the three element case, the summed response had a period one third that of each element.

Now to determine the critical value analytically, as the oscillations begin once the bifurcation point has been reached, I simply need to calculate the critical K at which at least one of the eigenvalues of each of the eight fixed points equals zero [1]. This results in a confirmation of the numerical approximation with $K_c \simeq 0.546918$, also reciprocated in numerical simulations in MATLAB. Unfortunately, no exact solution is possible due to the complexity of the eigenvalues.

2.2 Relationship between ν and K

In order to determine how the frequency of the oscillations and the coupling strength are related, I make use of the decoupling method devised by V. In *et al.* [4].

From Figure 5, it can be seen that while one element is switching states, the other is mostly stationary, and can hence be approximated as staying in the same state. This property is crucial for the decoupling method and is present in the system for K values close to the critical point. Therefore, this analytical method can be used for small perturbations of K.

Supposing element 1 to be the element transitioning from positive to negative, in which case element 2 is initially in its positive state, decoupling the system results in the following

$$f(x_1) = \dot{x_1} = (1+K)x_1 - (x_1)^3 - Kx_{2m} \tag{7}$$

where x_{2m} is the value of x_2 in its positive state.

Now that x_1 is decoupled from x_2, its behaviour can be expressed with a potential function (8)

$$U_m(x_1) = -\frac{1}{2}(1+K)(x_1)^2 + \frac{1}{4}(x_1)^4 + Kx_{2m}x_1 \tag{8}$$

At the critical coupling, this potential loses its bistability at which point $U'_m(x_1) =$

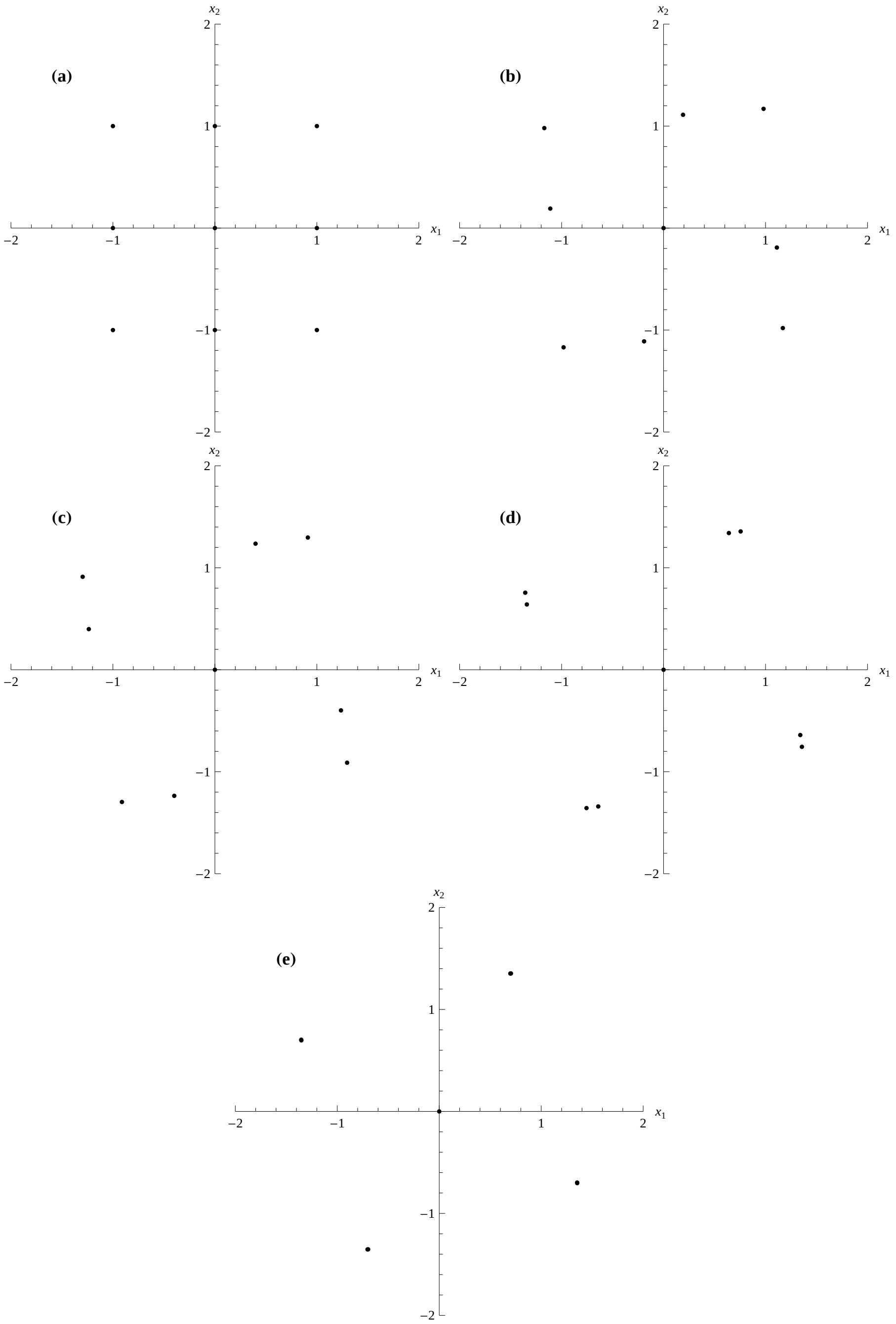

Figure 4: The fixed points in phasespace for the unforced system at (a) $K = 0.0001$, (b) $K = 0.2$, (c) $K = 0.4$, d) $K = 0.5395$ and (e) $K_c = 0.5469$.

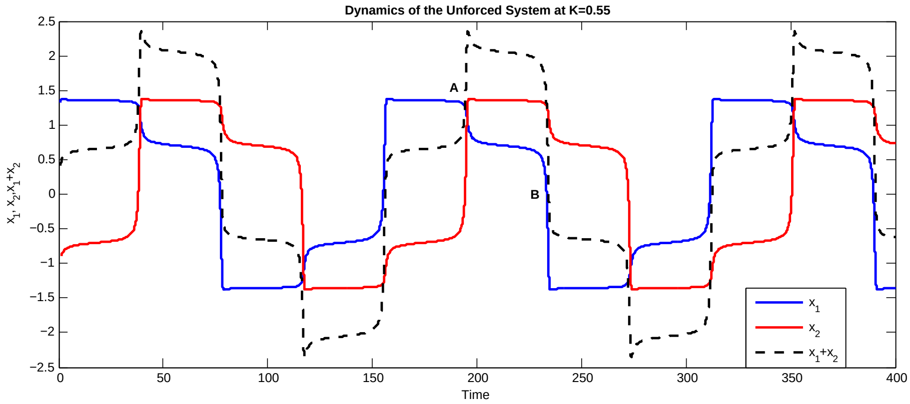

Figure 5: The oscillations of the unforced system at K =0.55.

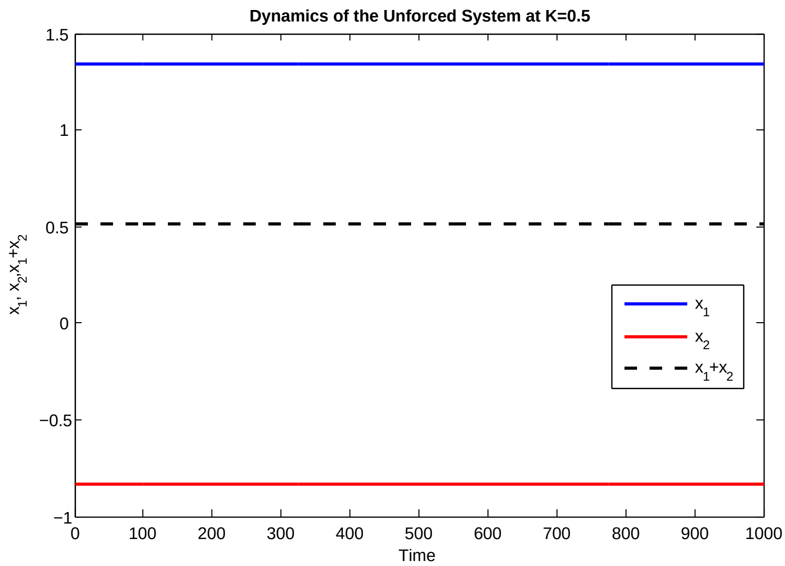

Figure 6: One of the steady states of the unforced system at K =0.5.

$U''_m(x_1) = 0$ where $'$ denotes the derivative with respect to x_1.

$$\begin{aligned} U''_m(x_1) &= -(1+K) + 3(x_1)^2 = 0 \\ \Rightarrow x_{1c} &= \sqrt{\frac{1+K_c}{3}} \end{aligned} \tag{9}$$

$x_1 > 0$ as it is in the positive state. Substitution into $U'_m(x_1) = 0$ yields

$$K_c x_{2m} = \frac{2}{3}\frac{(1+K_c)^{\frac{3}{2}}}{\sqrt{3}} \tag{10}$$

Then, as element 2 is constant in its positive state, $\dot{x_2} = 0$ and substituting $x_1 = \sqrt{\frac{1+K_c}{3}}$ gives for $x_{2m} > 0$

$$\begin{aligned} x_{2m} &= \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}(1+K)}{\left(3\sqrt{3}K\sqrt{1+K} + \sqrt{3}\sqrt{-4-12K-3K^2+5K^3}\right)^{\frac{1}{3}}} \\ &+ \frac{\left(3\sqrt{3}K\sqrt{1+K} + \sqrt{3}\sqrt{-4-12K-3K^2+5K^3}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}3^{\frac{2}{3}}} \end{aligned} \tag{11}$$

Using these solutions, equation (7) can be integrated from point A to point B ($x_1 = 0$) in Figure 5 giving the time required for the elements to switch state; this gives a quarter of the period of the summed signal assuming that the time needed to move from the "zero" state to the negative state is negligible and using the symmetry of the system.

$$t = \int_{x_{+m}}^{0} \frac{1}{f(x_1)}\,dx_1 = -\int_{0}^{x_{+m}} \frac{1}{f(x_1)}\,dx_1$$

where x_{+m} is the positive steady state for x_1. Noting that $x_{10} = \sqrt{\frac{1+K}{3}}$ is the point where the integrand $-\frac{1}{f(x_1)}$ is maximised, taking a Taylor expansion of $f(x_1)$ about x_{10} and using algebraic tricks obtained from V. In *et al.* [4] gives

$$t \simeq \int_{-\infty}^{\infty} \frac{1}{f(x_{10}) + \frac{f''(x_{10})x^2}{2}}\,dx$$

$$t \simeq \frac{\sqrt{2}\pi}{\sqrt{f(x_{10})\,f''(x_{10})}} \tag{12}$$

From symmetry, the period of the summed signal is $4t$ giving the frequency,

$$\nu = \frac{1}{4t}$$

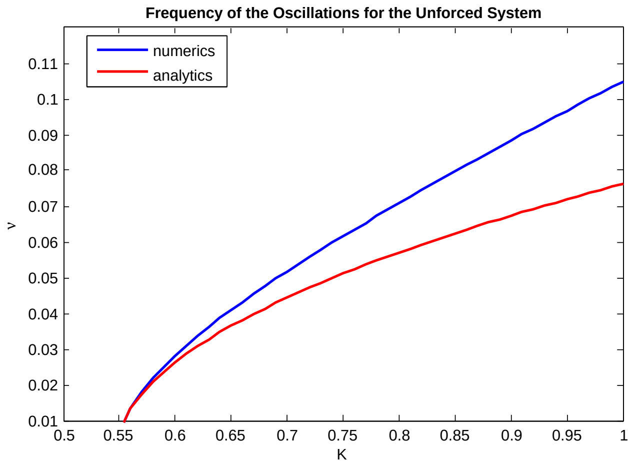

Figure 7: The frequency of the oscillations for the unforced model against the coupling strength, both numerical simulations and analytical approximations.

$$\nu \simeq \frac{\left(\sqrt{\left(k\left(-6k+2k^3-3K\left(2k-\frac{2^{\frac{4}{3}}k^2}{(3kK+s)^{\frac{1}{3}}}-2^{\frac{2}{3}}\left(3kK+s\right)^{\frac{1}{3}}\right)\right)\right)}\right)}{4\sqrt{6}\pi} \quad (13)$$

$$\text{where } k = \sqrt{1+K}$$
$$s = \sqrt{(-2+K)(1+K)(2+5K)}$$

For small perturbations of K from K_c, expanding (14) about $K_c = 0.5469$ gives

$$\nu \simeq \frac{\sqrt{2.025K - 1.106}}{4\pi}$$

Numerical simulation via MATLAB gives the relationship between the coupling strength and the oscillation frequency shown in Figure 7 with a comparison to the analytical relationship. It can be seen that the analytics give a close approximation only for values of K close to K_c as the assumptions made in the calculations are only valid close to this critical value.

3 Dynamics of the system in response to a constant input

A logical place to begin when observing the changes of the system in the presence of an input signal is with a small constant signal, $\epsilon > 0$. This can be added to the system in different ways and I will be investigating two such ways. The first I will call the 'alternating' configuration and the second the 'matching' configuration. The reason for these two configurations being of interest is the differing coupling setup in comparison with previous research [2, 4]. The two-element system has not been studied before and it is possible that with the negative coupling, the input could be modelled in two different ways, hence for completeness, both configurations must be studied. Note for all numerical simulations, $\epsilon = 0.01$, unless otherwise stated.

3.1 The alternating system

For the alternating configuration, the new differential equations modelling the system are as follows

$$\begin{aligned} \dot{x_1} &= x_1 - (x_1)^3 + K(x_1 - x_2) + \epsilon \\ \dot{x_2} &= x_2 - (x_2)^3 + K(x_1 + x_2) - \epsilon \end{aligned} \tag{14}$$

where for element 2, the input is modeled with "-ϵ". These differential equations yield the new potentials (15) and (16) which, as shown in Figures 8 and 9, are no longer symmetric and instead are biased respectively towards the positive and negative wells.

$$U(x_1) = -\frac{1}{2}(x_1)^2 + \frac{1}{4}(x_1)^4 - \epsilon x_1 \tag{15}$$

$$U(x_2) = -\frac{1}{2}(x_2)^2 + \frac{1}{4}(x_2)^4 + \epsilon x_2 \tag{16}$$

This bias indicates that the positive state is favoured over the negative state for x_1 and vice versa for x_2. To analyse the new dynamics of this system, I will use similar techniques as in the unforced case.

3.2 Numerical simulation and fixed point analysis (alternating system)

The previous method of obtaining an estimate of the critical coupling through the numerical analysis of the fixed points using Mathematica to solve the simultaneous equations (17) unfortunately produces very complex solutions which cannot be reliably plotted.

$$\begin{aligned} x_1 - (x_1)^3 + K(x_1 - x_2) + 0.01 &= 0 \\ x_2 - (x_2)^3 + K(x_1 + x_2) - 0.01 &= 0 \end{aligned} \tag{17}$$

I have roughly approximated the critical K values instead by viewing a 3D graph in Mathematica of the differential equations with the Manipulate function to see how the intersection between the surfaces and the zero-plane changes with K. From the estimated critical K's obtained, I was able to produce several fixed points plots, see Figure 10, by

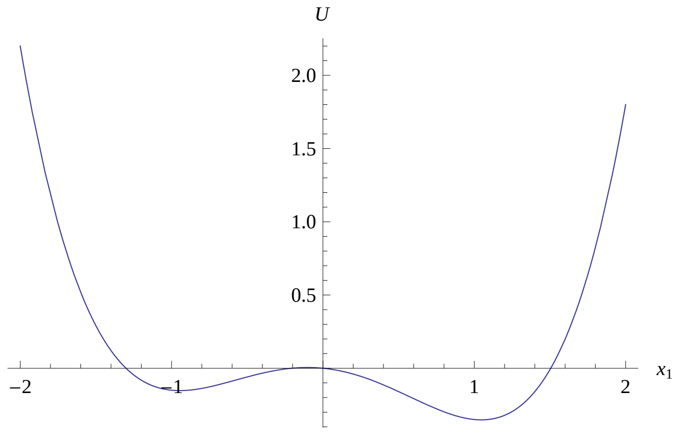

Figure 8: The bistable potential for x_1 with $\epsilon = 0.1$. This value of ϵ was chosen to emphasize the bias towards the positive well.

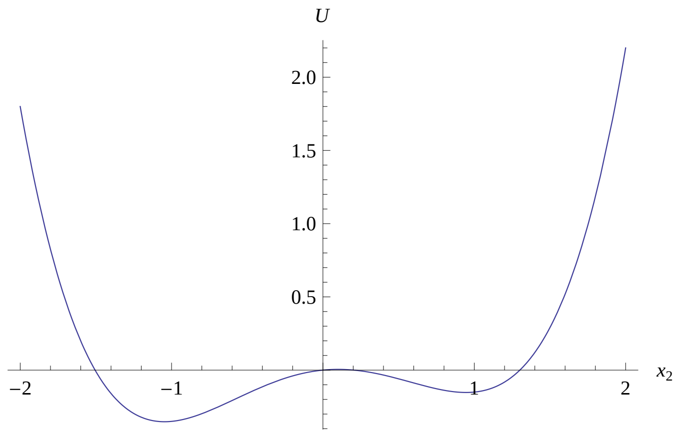

Figure 9: The bistable potential for x_2 with $\epsilon = 0.1$ showing the bias towards the negative well.

solving (17) numerically around those approximations, and as can be seen, there is more than one critical K value. This can be expected as the potentials are no longer symmetric and hence, it will be easier, i.e. a lower K is required, to 'destroy' certain pairs of fixed points before the others. In the case of x_1, as the positive state is favoured, fixed points with negative x_1 are expected to be destroyed first and the opposite is true for x_2. Also taken into account is the preference of x_1 to be in the opposite state to x_2 and the initial conditions of the system. The dynamics of the system can be seen in Figure 11. Again as the fixed points disappear in pairs, saddle node bifurcations are taking place.

Performing a numerical simulation in MATLAB results in the approximate critical K value of 0.55903. Note that this value is larger than the critical K of the unforced system. As the oscillations can only begin once all the stable states have disappeared, the value of K required must be larger than that required to exit the state to which the system is biased, for example for element 1, the K must exceed the value to eliminate the positive well in the bistability destruction.

3.3 Relationship between ν and K (alternating system)

The decoupling method can again be applied to determine the frequency-coupling relationship even with the addition of the constant input as demonstrated in [4]. As will be shown later, the only change required is in the final integration for calculating the period of the oscillations. As the constant signal removes the symmetry of the bistable potential, and hence the system dynamics, the transition from positive to negative actually takes longer than from negative to positive for element 1 and vice-versa for element 2. Hence two different integrals must be evaluated to determine the period. The decoupled differential equation and corresponding potential for x_1 during transition is as follows where x_{2m} is again the value of x_2 in the positive state.

$$f(x_1) = \dot{x_1} = (1+K)x_1 - (x_1)^3 - Kx_{2m} + \epsilon$$

$$U_m(x_1) = -\frac{1}{2}(1+K)(x_1)^2 + \frac{1}{4}(x_1)^4 + Kx_{2m}x_1 - \epsilon x_1$$

Again the critical coupling occurs when the potential loses its bistability ($U'_m(x_1) = 0$ and $U''_m(x_1) = 0$) giving

$$\begin{aligned} x_{1c} &= \pm\sqrt{\frac{1+K_c}{3}} \\ K_c x_{2mc} &= \frac{2}{3}\frac{(1+K_c)^{\frac{3}{2}}}{\sqrt{3}} + \epsilon \end{aligned} \tag{18}$$

Now under the approximation that ϵ is small, following the procedure outlined in V. In *et al.* [4] for a small constant signal, to calculate x_{2m} and K_c, the following is assumed

$$\begin{aligned} K_c &= K_{c0} + \delta_1 \\ x_{2m} &= x_{2m0} + \delta_2 \end{aligned} \tag{19}$$

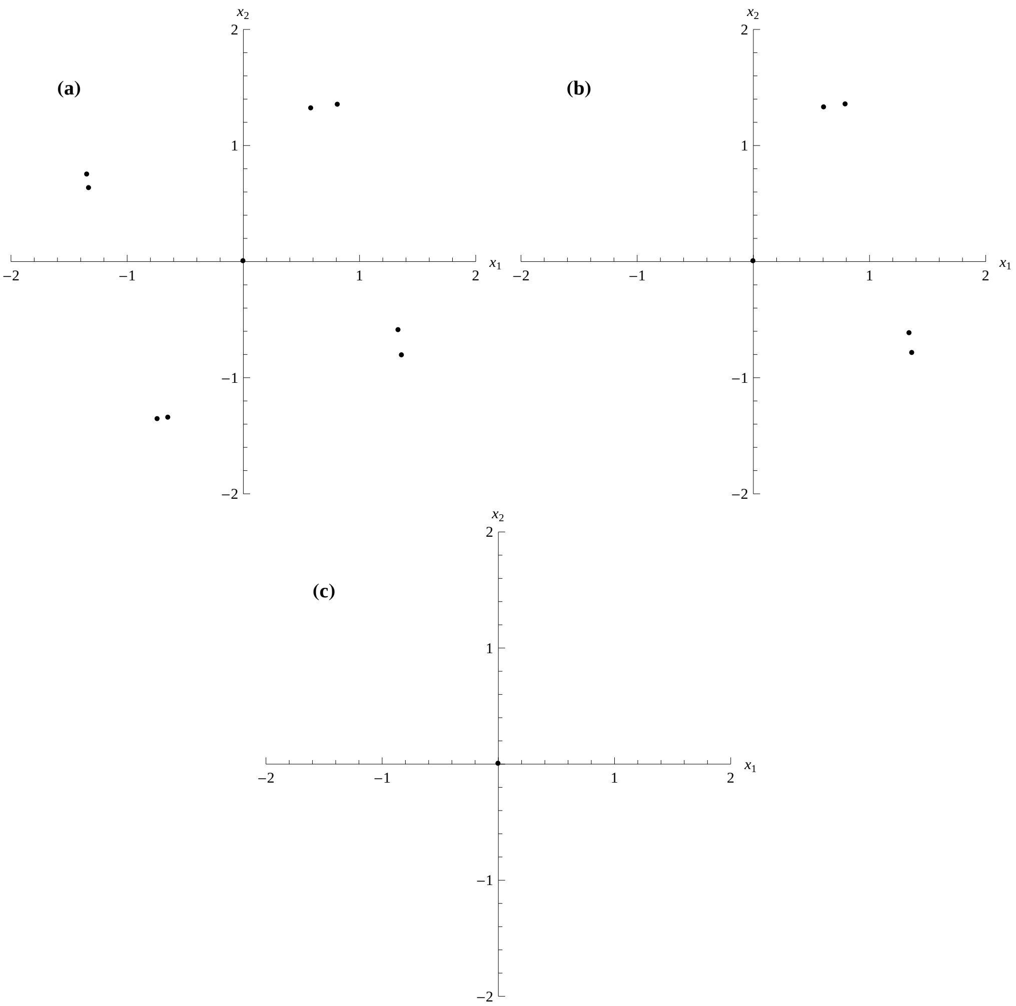

Figure 10: The fixed points in phasespace for the alternating system at (a) $K = 0.53$, (b) $K = 0.54$ and (c) $K = 0.56$.

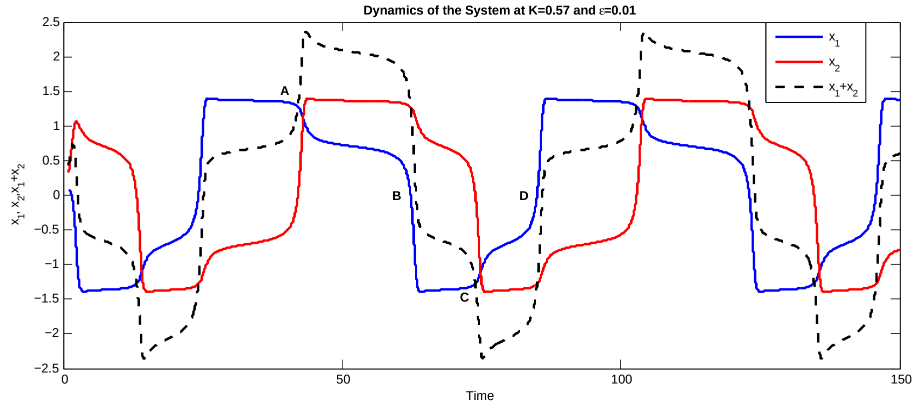

Figure 11: The oscillations of the alternating system at K=0.57. Note the different speeds of transition.

where K_{c0} and x_{2m0} are the values of K_c and x_{2m} from the unforced case and the $\delta's$ are small.

Again during the transitioning of element 1, element 2 is approximately stationary giving

$$\begin{aligned} \dot{x_2} = f(x_{2m}) = 0 &= (1+K)x_{2m} - (x_{2m})^3 + Kx_1 - \epsilon \\ &= (1+K)x_{2m} - (x_2m)^3 + K\sqrt{\frac{1+K}{3}} - \epsilon \end{aligned} \quad (20)$$

where $x_1 = \sqrt{\frac{1+K}{3}}$ was substituted from (18). Substituting (19) into (20) and taking δ_2 to first order yields

$$\begin{aligned} \delta_2 &= -\frac{\epsilon}{1+K} \\ \Rightarrow x_{2m} &= x_{2m0} - \frac{\epsilon}{1+K} \end{aligned} \quad (21)$$

Similarly, for the transition of element 1 from the negative state to the positive,

$$\begin{aligned} x_{101} = -\sqrt{\frac{1+K}{3}} \\ \delta_{22} &= \frac{\epsilon}{1+K} \\ \Rightarrow x_{2m2} &= x_{2m02} + \frac{\epsilon}{1+K} \end{aligned} \quad (22)$$

where $x_{2m02} = -x_{2m0}$.

In order to determine δ_1, the first of (19) must be substituted into the second of (21) and both then substituted into 18 before taking δ_1 to first order. However, as the formulas obtained for x_{2m0} and K_c were so complicated, no easily expressible solution was found.

Now to calculate the period, as noted before, inclusion of the offset destroys the symmetry of the system causing certain transitions to take shorter times for completion. However, it will be assumed that each element has the same period which is also the period of the summed response. Thus, two integrations must be done; one for the transition of element 1 from the positive to negative state (from point A to point B in Figure 11) and the other from the negative to positive state (point C to point D). For the positive to negative transition, using the same method as in the unforced case,

$$\begin{aligned} t_1 &\simeq \int_{-\infty}^{\infty} \frac{1}{f\left(x_{10}\right)+\frac{f''\left(x_{10}\right)x^2}{2}}\,dx \\ &\simeq \frac{\sqrt{2}\pi}{\sqrt{f\left(x_{10}\right)f''\left(x_{10}\right)}} \end{aligned} \tag{23}$$

where $x_{10}=\sqrt{\frac{1+K}{3}}$ and substituting x_{2m} from previous calculation. And for the negative to positive transition,

$$\begin{aligned} t_2 &\simeq \int_{-\infty}^{\infty} \frac{1}{f\left(x_{101}\right)+\frac{f''\left(x_{101}\right)x^2}{2}}\,dx \\ &\simeq \frac{\sqrt{2}\pi}{\sqrt{f\left(x_{101}\right)f''\left(x_{101}\right)}} \end{aligned} \tag{24}$$

where $x_{101}=-\sqrt{\frac{1+K}{3}}$ and $x_{2m2}=x_{2m}$ in the formula for f.

This gives a total period of $t=2\left(t_1+t_2\right)$, assuming the time from point A to B equals the time from D to A (of the next oscillation cycle) and the time from B to C equals that from C to D, resulting in a frequency of

$$\nu=\frac{1}{t}=\frac{1}{2t_1+2t_2}$$

Numerical simulations in MATLAB yields the relationship between the frequency of oscillation and the coupling strength, as shown in Figure 12.

3.4 The matching system

In this configuration, the constant input is modelled with "$+\epsilon$" for both elements yielding equations

$$\begin{aligned} \dot{x_1} &= x_1-\left(x_1\right)^3+K\left(x_1-x_2\right)+\epsilon \\ \dot{x_2} &= x_2-\left(x_2\right)^3+K\left(x_1+x_2\right)+\epsilon \end{aligned} \tag{25}$$

They now share the same potential (15) and hence the same bias shown previously in Figure 8.

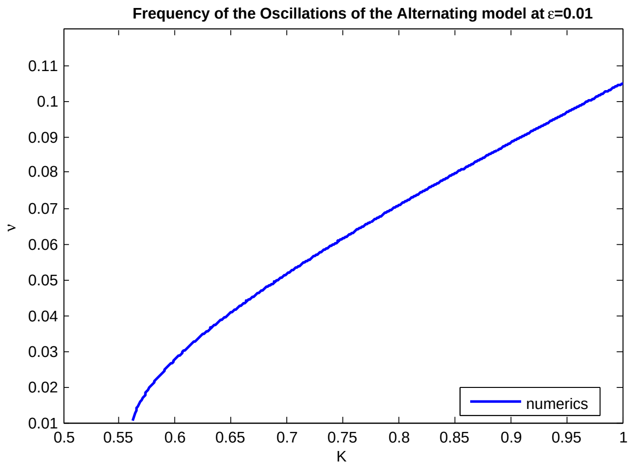

Figure 12: The frequency of the oscillations for the alternating model against the coupling strength.

3.5 Numerical simulation and fixed point analysis (matching system)

Using the same procedure as in the alternating case, solutions of the simultaneous equations (26) are plotted, see Figure 13, for various K values obtained from viewing the 3D graph.

$$\begin{aligned} x_1 - (x_1)^3 + K(x_1 - x_2) + 0.01 &= 0 \\ x_2 - (x_2)^3 + K(x_1 + x_2) + 0.01 &= 0 \end{aligned} \tag{26}$$

In this case, both elements favour the positive state leading to the preferred earlier destruction of the negative states. Again, the preference of element 1 to be in the opposite state and the initial conditions affect the order of disappearance and their disappearance in pairs signifies saddle node bifurcations. Figure 14 shows the behaviour of the system at K=0.57. Notice how both elements take a longer time to transition from the positive state to the negative state but spend different amounts of time in each state. Each element must balance between the bias towards the positive state and the preferred state as a result of the coupling. Performing a numerical simulation in MATLAB results in the approximate critical K value of 0.55607; larger than the unforced critical K for the same reasons as in the alternating case.

3.6 Relationship between ν and K (matching system)

Once again using the decoupling method from V. In *et al.* [4], as in the alternating configuration, the integration must be completed in two sections. From Figure 14, it will be assumed from symmetry that the time from point A to B equals that of B to C and the time from C to D equals that of D to A (of the next oscillation cycle).

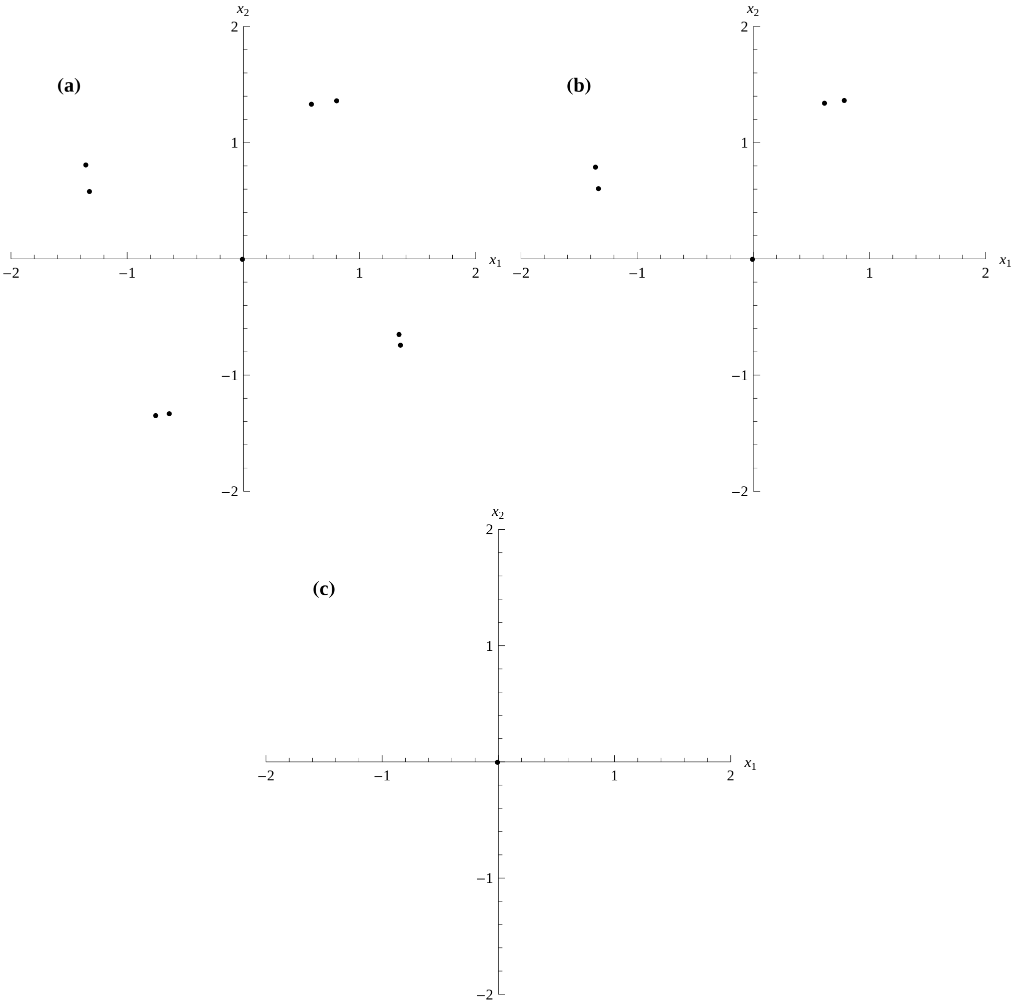

Figure 13: The fixed points in phasespace for the matching system at (a) $K = 0.53$, (b) $K = 0.54$ and (c) $K = 0.56$.

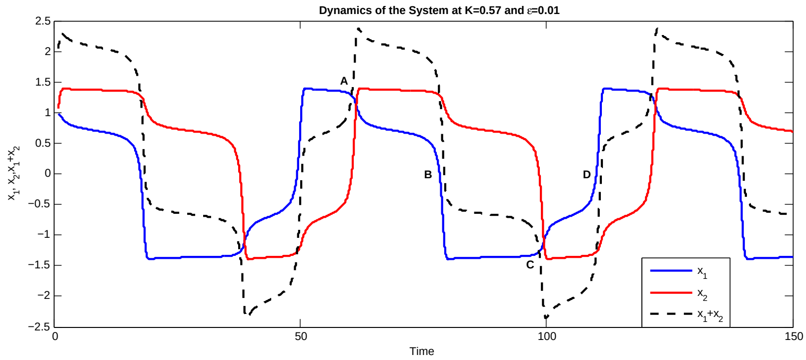

Figure 14: The oscillations of the matching system at K=0.57. Note the different speeds of transition.

Both elements and the summed response again have the same period. As before, during the transitioning of element 1, element 2 is approximately stationary allowing for the decoupling of element 1 from element 2 giving

$$f(x_1) = \dot{x_1} = (1+K)x_1 - (x_1)^3 - Kx_{2m} + \epsilon$$

$$U_m(x_1) = -\frac{1}{2}(1+K)(x_1)^2 + \frac{1}{4}(x_1)^4 + Kx_{2m}x_1 - \epsilon x_1$$

Again the critical coupling occurs when the potential loses its bistability giving

$$\begin{aligned} x_{1c} &= \pm\sqrt{\frac{1+K_c}{3}} \\ K_c x_{2mc} &= \frac{2}{3}\frac{(1+K_c)^{\frac{3}{2}}}{\sqrt{3}} + \epsilon \end{aligned} \tag{27}$$

as before. Substitution into $\dot{x_2}$ gives

$$\begin{aligned} \dot{x_2} = f(x_{2m}) = 0 &= (1+K)x_{2m} - (x_{2m})^3 + Kx_1 + \epsilon \\ &= (1+K)x_{2m} - (x_2m)^3 + K\sqrt{\frac{1+K}{3}} + \epsilon \end{aligned} \tag{28}$$

Again assuming the following,

$$\begin{aligned} K_c &= K_{c0} + \delta_1 \\ x_{2m} &= x_{2m0} + \delta_2 \end{aligned} \tag{29}$$

where K_{c0} and x_{2m0} are the values of K_c and x_{2m} from the unforced case and the $\delta's$ are small. Performing the same calculations as before leads to the solution of δ_2

$$\delta_2 = \frac{\epsilon}{1+K} \tag{30}$$

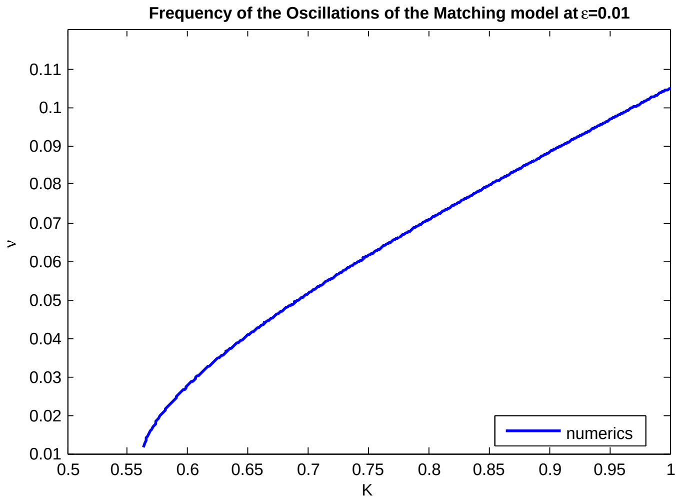

Figure 15: The frequency of the oscillations for the matching model against the coupling strength.

for both transitions from positive to negative and back again.

Again, no simple analytical solution for δ_1 is possible so a formula relating the frequency of oscillation and coupling is not explicitly expressible. The frequency however, would be calculated in the same manner as before. Numerical simulations in MATLAB give the relationship between coupling strength and oscillation frequency shown in Figure 15.

4 Dynamics of the system in response to noise

Another logical situation to be studied is the addition of noise to the model. Following the procedure used in the inclusion of a constant signal to the system, the noise will be added, with the use of a Wiener process [3]. The noise studied is Gaussian white noise with stochastic term $\mathrm{d}W = \sqrt{2D}\xi_i(t)\sqrt{\Delta t}$, where D is the variance of the noise and ξ is a Gaussian random variable with mean 0 and standard deviation 1.

4.1 Behaviour of the matching noise system

In the matching configuration, the stochastic differential equations are as follows,

$$\begin{aligned} \dot{x_1} &= x_1 - (x_1)^3 + K(x_1 - x_2) + \sqrt{2D}\xi_1(t) \\ \dot{x_2} &= x_2 - (x_2)^3 + K(x_1 + x_2) + \sqrt{2D}\xi_2(t) \end{aligned} \tag{31}$$

which will be investigated numerically using the Euler forward integration method. With this method however, the stochastic component of the differential equations, i.e. the noise, is instead scaled with the square root of the time step [3] giving

$$\begin{aligned} x_1(t+\Delta t) &= x_1(t) + \Delta t\left(x_1(t) - (x_1(t))^3 + K(x_1(t) - x_2(t))\right) + dW_1 \\ x_2(t+\Delta t) &= x_2(t) + \Delta t\left(x_2(t) - (x_2(t))^3 + K(x_2(t) + x_1(t))\right) + dW_2 \end{aligned} \tag{32}$$

MATLAB simulations yield a critical coupling strength of $K_c \simeq 0.542$. This will however not always be the case as the noise strength is random, however providing the noise stays about the same order of magnitude, this value will be a close approximation.

As opposed to the constant signal cases, this critical coupling value is smaller than that of the unforced case. This is a result of the random nature of the noise. As the noise strength can vary, occasionally when the coupling strength is close to the critical point, a change in the noise can cause the coupling strength required to drop below the current K value therefore allowing for an earlier onset of oscillation than before. The oscillations are shown in Figure 16 and the frequency vs coupling strength relationship is shown in Figure 17.

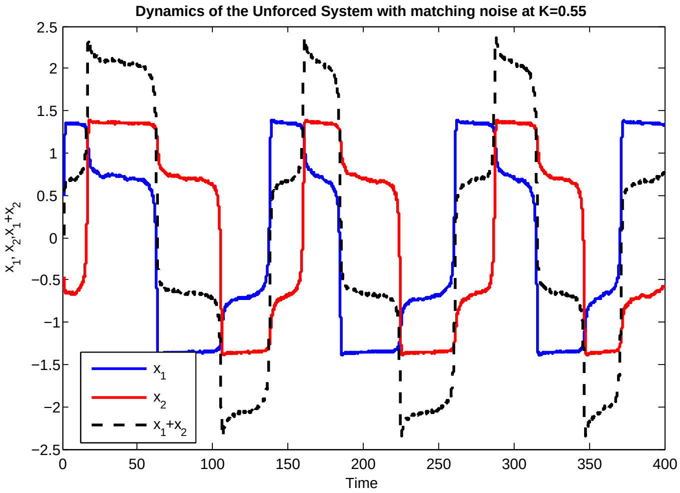

Figure 16: The oscillations for the unforced system with matching noise at K=0.55.

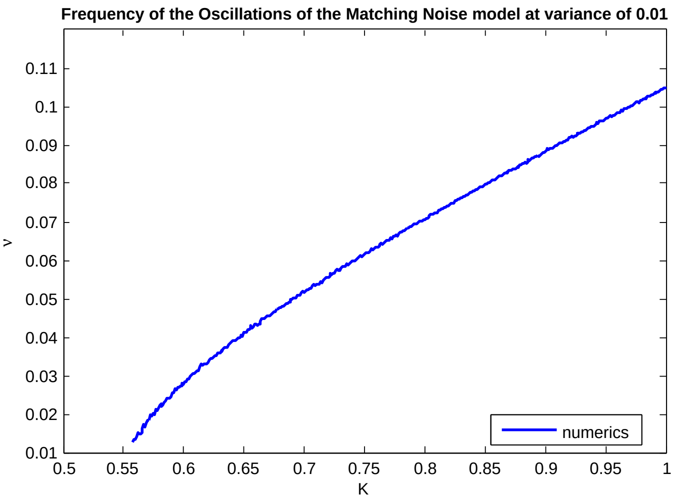

Figure 17: The frequency of the oscillations for the unforced system with matching noise against the coupling strength.

5 Conclusion

What this study has shown is ring systems with an even number of elements can begin to oscillate once the coupling strength has exceeded the threshold, provided there is a mix of the positive and negative coupling. This and the system's response to simple input signals together with its relative stability in the presence of noise makes its use as a signal detector viable.

The input of small positive constant signals caused the oscillations to become asymmetric where each element spent different amounts of time in each state in comparison to the unforced case, and also took different times to transition between states. These easily visible differences are amplifications of the response to a small signal and therefore reliably indicate the presence of a signal. The strength of the signal can be measured by the extent to which these times differ from the unforced response.

These systems of elements also do not require much power to run making them more efficient and economically attractive. The point of maximum sensitivity of the system, soon after the bifurcation point when the oscillations begin, can be taken advantage of if the coupling strength between the elements can be controlled. Another benefit is the relative lack of initial condition requirements. These systems merely require a frustration to be present in the ring. This makes it useful for use in practice as initial conditions are usually very difficult to control.

The addition of noise to the system, while causing the oscillations to lose their symmetry, does not have much of an impact on the frequency trend, other than the occasional outlier. This implies the system is relatively resilient to noise increasing its suitability as a signal detector.

The two-element system has some benefits over the three-element system in that there are less elements making the ring cheaper and simpler to construct. That less elements are present also reduces the likelihood of mechanical failure, however there are some benefits to the three-element system. As can be seen in previous research [4, 2], the analytical formulae for calculating the critical coupling strength and the frequency - coupling strength relationship are much more simple which means the measurement of input signal strengths is easier and more accurate. Further studies on this subject could cover the response of the system to periodic signals, i.e. sinusoidal signals as the main electricity form used is AC and whether synchronisation occurs where the frequency of the oscillations match the frequency of the signal, as well as the possibility of stochastic resonance when noise is added to the system. Also useful would be a general extension of the analytical study of this system to achieve more accurate formulas and hence a better understanding of the systems behaviour.

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