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An Investigation of the Homotopy Analysis Method for solving non-linear differential equations

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Abstract

Many problems which arise in applied mathematics are highly non-linear and thus can be difficult or impossible to solve analytically. The Homotopy Analysis Method (HAM) is a semi-analytical technique used to solve differential equations, in particular non-linear and partial. The technique utilises homotopy (the concept of deforming one continuous equation into another) in order to generate a convergent series of linear equations from non-linear ones. HAM was first proposed by Shijun Liao of Shanghai Jiaotong University in 1992 and since then has been widely implemented for solving non-linear differential equations in areas ranging from science to finance. In this paper we go about comparing HAM with other previously well established methods for solving differential equations including the Taylor series and Padé approximation by solving a non-linear differential equation proposed by Liao with each method. This was done in order to gain a better understanding of the effectiveness of HAM and determine if the method is worthy of its acclaim.

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1 Homotopy Analysis Method (HAM)

HAM is an analytical technique used to solve non-linear/partial differential equations by using homotopy (deformation of one continuous function into another) to generate a series of convergent linear equations from a non-linear one. HAM was first developed in 1992 by Liao and was further modified in 1997 to include the auxiliary parameter $\hbar$. This non-zero parameter allows control over the convergence of the series. Because HAM is based off the concept of homotopy we have the freedom to choose the initial approximation of the solution, the auxiliary linear operator and the convergence control parameter, $\hbar$. This property sets HAM apart from other methods as we can choose the equation-type of the high order deformation equation and the base functions of its solution.

1.1 Basic Idea

Here we give a brief derivation of HAM for solving a non-linear differential equation. If we have the non-linear differential equation:

$$\mathcal{N}[f(t)] = 0 \tag{1}$$

where $\mathcal{N}$ is a non-linear operator, t is the independent variable and $f(t)$ is an unknown function. We can construct the homotopy,

$$H[\phi(t;q), f_0(t), \hbar, q] = (1-q)L[\phi(t;q) - f_0(t)] - q\hbar H(t)N[\phi(t;q)] \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $\hbar$ is a non-zero auxiliary parameter, $H(t)$ is an auxiliary function ($H(t) \neq 0$), $\mathcal{L}$ is an auxiliary linear operator, $f_0(t)$ is an initial approximation of $f(t)$ (which satisfies the initial conditions) and $\phi(t;q)$ a function which also must satisfy the initial conditions. Setting the homotopy equal to zero, we construct the zero-th order deformation equation:

$$(1-q)\mathcal{L}[\phi(t;q) - f_0(t)] = q\hbar H(t)\mathcal{N}[\phi(t;q)] \tag{3}$$

whose solution transforms continuously with respect to q. When $q = 0$ it follows that

$$\mathcal{L}[\phi(t;0) - f_0(t)] = 0 \tag{4}$$

Now from the definitions of $\mathcal{L}[f(t)]$, $\phi(t;q)$ and $f_0(t)$

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$$\phi(t;0) = f_0(t) \tag{5}$$

Similarly when $q = 1$,

$$\mathcal{N}[\phi(t;1)] = 0 \tag{6}$$

Since $\phi(t;q)$ must satisfy the initial conditions of the differential equation, from 6 it follows that $\phi(t;1) = f(t)$. From this we can see that $\phi(t;q)$ varies continuously from the initial approximation $f_0(t)$ to the solution as the q increases from 0 to 1. As previously stated, we wish to take the non-linear equation and form a series of linear convergent approximations. We define the mth order linear approximation $f_m(t)$ given as

$$f_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}\bigg|_{q=0} \tag{7}$$

which is called the mth-order deformation derivative. Now we can expand $\phi(t;p)$ using the Taylor series with respect to q:

$$\phi(t;q) = \phi(t;0) + \sum_{m=1}^{\infty} \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}\bigg|_{q=0} q^m \tag{8}$$

Thus using equation 7 and $\phi(t;0) = f_0(t)$, the power series becomes

$$\phi(t;q) = f_0(t) + \sum_{m=1}^{\infty} f_m(t) q^m \tag{9}$$

If the auxiliary linear operator, initial guess, $\hbar$ and the auxiliary function are chosen so the series converges at $q = 1$,

$$f(t) = f_0(t) + \sum_{m=1}^{\infty} f_m(t) \tag{10}$$

We now go about finding an expression for the $f_m(t)$ by differentiating equation 3 with respect to q:

$$(1-q)\left(\frac{\partial \phi(t;q)}{\partial q}\right) - \mathcal{L}(\phi(t;q) - f_0(t)) = \hbar H(t)\mathcal{N}[\phi(t;q)] + q\hbar H(t)\frac{\partial \mathcal{N}[\phi(t;q)]}{\partial q} \tag{11}$$

By setting $q = 0$ and again using equation 7,

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$$\mathcal{L}[f_1(t)] = \hbar H(t)\mathcal{N}[f_0(t)] \tag{12}$$

That is, the linear value of $f_1(t)$ can be obtained from a non-linear transformation of $f(t)$. By extending this process we can get

$$\mathcal{L}[f_m(t) - f_{m-1}(t)] = \hbar H(t)\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0} \tag{13}$$

By rearranging this equation and introducing the term $\chi(m)$

$$\chi_m = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 & \text{if } n > 1 \end{cases} \tag{14}$$

we can form the mth order deformation equation:

$$\mathcal{L}[f_m(t) - \chi(m)f_{m-1}(t)] = \hbar H(t)\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0} \tag{15}$$

which is true for $m \geq 1$. Rearranging

$$\mathcal{L}[f_m(t)] = \chi(m)f_{m-1}(t) + \mathcal{L}^{-1}\left\{\hbar H(t)\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0}\right\} \tag{16}$$

where $\mathcal{L}^{-1}$ is the inverse of the linear operator (i.e. inverse of differentiation is integration). The solution to $u(t)$ can be expressed as

$$f(t) = \sum_{m=0}^{\infty} f_m(t) \tag{17}$$

which is valid where ever the solution converges.

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2 Taylor Series Approximation (BEEF)

The *Taylor series* is a representation of a function as a power series whose terms are calculated from the function's derivatives at a particular point. By combining a *Mclaurin series*, a special case of the Taylor series centred at 0, with another Taylor series we can form a series solution of an ODE.

We consider the initial value problem (IVP) $f'(t) = y$ with the initial condition $f(0) = f_0$. First we go about finding an approximation for $f(t)$ centred at $a = 0$. First we form the set of derivatives by taking up to the mth derivative of $f(t)$

$$\left\{f'(t), f''(t), f'''(t), \ldots, f^{(m)}(t)\right\} \tag{18}$$

By evaluating each derivative at $t = 0$ using the initial value f_0, we then can form the set

$$\left\{f_0, f'(0), f''(0), f'''(0), \ldots, f^{(m)}(0)\right\} \tag{19}$$

Using this set of derivatives and we can form a series solution for $f(t)$ using the Mclaurin series:

$$\begin{aligned} f(t) &= \sum_{i=0}^{m} \frac{f^{(i)}(0)}{i!}(t)^i \\ &= f_0 + f'(0)t + \frac{f''(0)}{2}t^2 + \ldots + \frac{f^m(0)}{m!}t^m \end{aligned} \tag{20}$$

While this series can give results of a reasonable accuracy around $t = 0$ when taking a sufficient number of terms, we may wish to centre the series around non-zero values. However this requires a value of $f(a)$ which we can approximate using another Taylor series. By taking out previously determined set of derivatives, we form:

$$\begin{aligned} f(a) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)a^n}{n!} \\ &= f(0) + f'(0)a + \frac{f''(0)a^2}{2} + \ldots \end{aligned} \tag{21}$$

Thus by taking the derivatives of $f(a)$ to get $f'(a), f''(a), \ldots$ we can substitute them into a general Taylor series to form a general solution for the BEEF method

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$$f(t) = \sum_{j=0}^{n} \sum_{i=0}^{m} \frac{f^{(i)}(0)a^i}{i!} (t-a)^j \tag{22}$$

when taking m terms of the inner series and n terms of the outer series.

If we consider the BEEF solution to be of polynomial form $f(t) = \sum_{i=0}^{\infty} c_i t^i$ then we have the expression for c_i when taking m outer series terms

$$c_i = \sum_{n=i}^{m} \frac{1}{n!} \binom{n}{n-i} f^{(n)}(a)(-a)^{n-i} \tag{23}$$

which will be useful when comparing the series solution of BEEF to that of the HAM solution.

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3 Padé Approximations

One method of improving the accuracy of a function approximation by using a rational function approximation. The *Padé approximate* is a method developed around 1890 by French mathematician Henri Padé used to give a better approximation given a truncated power series. The Padé approximate may converge even when the power series does not.

If we consider a Taylor polynomial of $f(t)$ truncated to m terms, $P_m(t)$, then

$$f(t) - P_m(t) = \mathcal{O}(x^{m+1}) \tag{24}$$

We define the rational function $R(t)$

$$R(t) = \frac{\sum_{i=0}^{m} a_i t^i}{1 + \sum_{j=1}^{n} b_j t^j} \tag{25}$$

and say $R(t)$ is of the degree $N = m + n$. Since $R(t)$ has $m + n + 1$ parameters then we can expect:

$$f(t) - R(t) = \mathcal{O}(t^{m+n+1}) \tag{26}$$

Since $R(t)$ is a 'richer class' of function than a polynomial (rational function with $q(t) = 1$), a Nth order Padé approximate will be at least as good as a Nth degree polynomial. Through careful selection of m and n we can expect this error to decrease. To find the coefficients a_i and b_j we equate $P_k(t)$ to $R(t)$ and solve the resulting system of linear equations where the coefficients of $P_m(t)$ are given by

$$c_i = \frac{f^{(i)}(0)}{i!} \tag{27}$$

so it follows that

$$c_0 + c_1 t + c_2 t^2 + ... = \frac{a_0 + a_1 t + a_2 t^2 + ... + a_n t_n}{1 + b_1 t + b_2 t^2 + ... + b_m t_m} \tag{28}$$

We can then equate coefficients where

$$\begin{aligned} a_0 &= c_0 \\ a_1 &= b_1 c_0 + c_1 \\ &\vdots \\ a_n &= b_m c_{n-m} + ... + b_1 c_{n-1} + c_m \end{aligned} \tag{29}$$

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$$
\begin{aligned}
c_{n+1} + b_1 c_n + ... + b_m c_{n-m+1} &= 0 \\
&\vdots \\
C_N + b_1 c_{N-1} + ... + b_m c_{N-m} &= 0
\end{aligned}
\tag{30}
$$

Thus forming the Padé approximation. A more specific example of this process is given in section 4.5.

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4 Non-linear example

Here we are going to be examining the non-linear ordinary differential equation as given in the 'Beyond Perturbation' book by Liao Shijun

$$\frac{dV}{dt} - V(t)^2 + 1 = 0 \tag{31}$$

with the initial conditions

$$V(0) = 0$$

Note that this has the exact solution, $V(t) = \tanh(t)$.

4.1 Forming the HAM solution

For simplicity we choose to represent the solution in the set of base functions

$$\{t^n | n = 1, 2, 3, ...\} \tag{32}$$

so the solution will be in the form

$$V(t) = \sum_{n=0}^{\infty} c_n t^n \tag{33}$$

where c_n is a coefficient which needs to be determined. From our ODE we also choose the linear operator

$$\mathcal{L}[\phi(t;p)] = \frac{\partial \phi(t;p)}{\partial t} \tag{34}$$

and our non-linear operator

$$\mathcal{N}[\phi(t;p)] = \frac{\partial \phi(t;p)}{\partial t} + \phi(t;p)^2 - 1 \tag{35}$$

We are free to choose the initial approximation so long as it satisfies the initial conditions so we use $V_0(t) = t$. Using these definitions we form the mth order deformation equation (given by equation 15)

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$$
\begin{aligned}
\mathcal{L}[N_m(t) - \chi N_{m-1}(t)] &= \hbar H(t) \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t;q)]}{\partial^{m-1} q} \\
&= \hbar H(t) \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial^{m-1} q} \left[\frac{\partial \phi(t;p)}{\partial t} + \phi(t;p)^2 - 1 \right] \\
&= \hbar H(t) \frac{1}{(m+1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} + \left[\sum_{n=0}^{\infty} V_m'(t) + \left(\sum_{n=0}^{\infty} V_n(t) q^n \right)^2 - 1 \right] \\
&= \hbar H(t) \left(V_{m-1}'(t) + \sum_{r=0}^{m-1} V_r(t) V_{m-r-1}(t) - (1 - \chi_{m-1}) \right)
\end{aligned} \tag{36}
$$

Now the inverse linear operator will be integration so we can form the expression

$$
V_m(t) = \chi_m N_{m-1}(t) + \hbar H(t) \int_0^t \left(V_{m-1}'(t) + \sum_{r=0}^{m-1} V_r(t) V_{m-r-1}(t) - (1 - \chi_{m-1}) \right) dt + C_m \tag{37}
$$

with C_m calculated from the initial conditions and the auxiliary function chosen as H(t) = 1. Using up to the 3rd order deformation equation with initial approximation $V_0(t) = t$, we have the HAM solution:

$$
V(t) = \frac{17\hbar^7}{315} t^7 + \left(\frac{2\hbar^3}{15} + \frac{4\hbar^2}{5}\right) t^5 + \left(\frac{\hbar^3}{3} + \hbar^2 + \hbar\right) t^3 + t \tag{38}
$$

4.2 Forming the BEEF solution

As stated previously, the BEEF solution is made up of two Taylor series, the 'inner' series which calculates $V(0), V'(0), V''(0) \ldots$ and the 'outer' series which uses the inner series to approximate $V(a)$. Here we show how to calculate the BEEF solution of the non-linear ODE. Suppose we choose to take the inner series to three terms where

$$
\begin{aligned}
V(0) &= V_0 = 0 \\
V'(0) &= 1 - V_0^2 = 1 \\
V''(0) &= 2V_0^3 - 2V_0^2 = 0
\end{aligned} \tag{39}
$$

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Thus we approximate $V(a)$ as:

$$V(a) \approx 0 + 1a + \frac{0a^2}{2!} = a \tag{40}$$

Thus using the outer Taylor series we can get an approximate of $V(t)$ as

$$\begin{aligned} V_1(t) &= a^3 + t(1-a^2) \\ V_2(t) &= a^3 + t^2(1-a^2) \\ V_3(t) &= a^5 + t(1+a^2-2a^4) + t^2(a^3-a) \end{aligned} \tag{41}$$

When $a = 0$, the BEEF solution produces the Maclaurin series of $\tanh(t)$:

$$V(t) = t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 ... \tag{42}$$

We note that when $\hbar = -1$, the HAM solution also gives this Maclaurin series.

4.3 Choosing auxiliary parameter $\hbar$

In 1997, the non-zero auxiliary parameter $\hbar$ was introduced to HAM which provides a family of expressions in terms of $\hbar$. The result of this is that the rate and region of convergence are dependent upon the value of $\hbar$. This apparently provides us with a convenient way to control the convergence of the HAM solution. According to Liao's book 'Beyond Perturbation', determining the optimal value of $\hbar$ involves plotting the $\hbar$-curves of the solution. By plotting the partial sums of $f_m(t)$ evaluated at a specific value of t against $\hbar$, we can expect to see the curve to be horizontal over the range for which the solution converges. For this specific problem, this range was found to be approximately $-1 \leq \hbar < 0$. To further refine this value, the HAM solution (30th order deformation) was plotted using different $\hbar$ values between -1 and 0 as seen in figure 1. Further experimentation showed $\hbar = -0.1$ was the optimal value for this particular IVP which was confirmed by Liao's book.

4.4 Comparing BEEF and HAM solutions

We go about comparing the BEEF and HAM (with $\hbar = -0.10$) solutions with the exact solution. Figure 2 compares the HAM solution taking the 30th order deformation with the Maclaurin series with 100 terms (BEEF centred at $a = 0$). The radius of convergence for the BEEF solution appears to be approximately 1.5 and we can show

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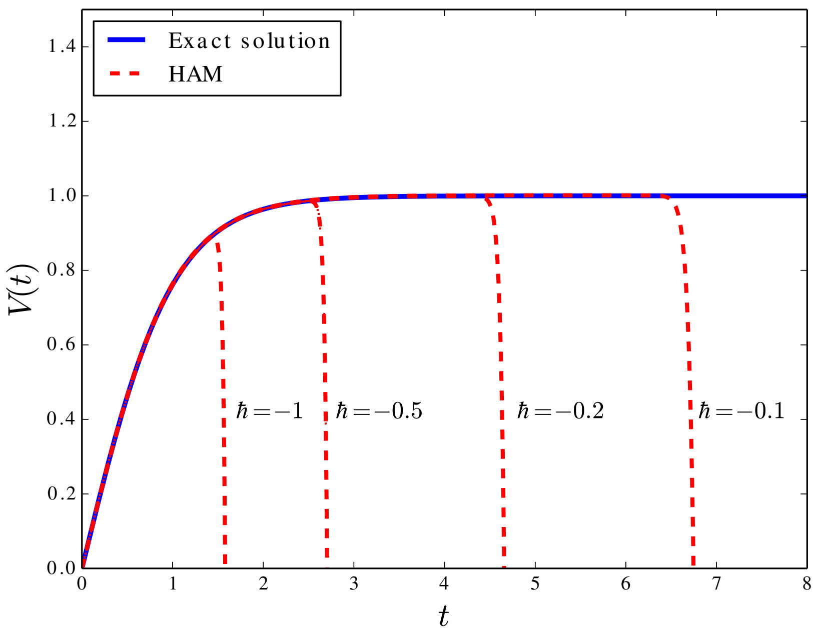

Figure 1: Comparison of various $\hbar$ values in order to determine its optimal value

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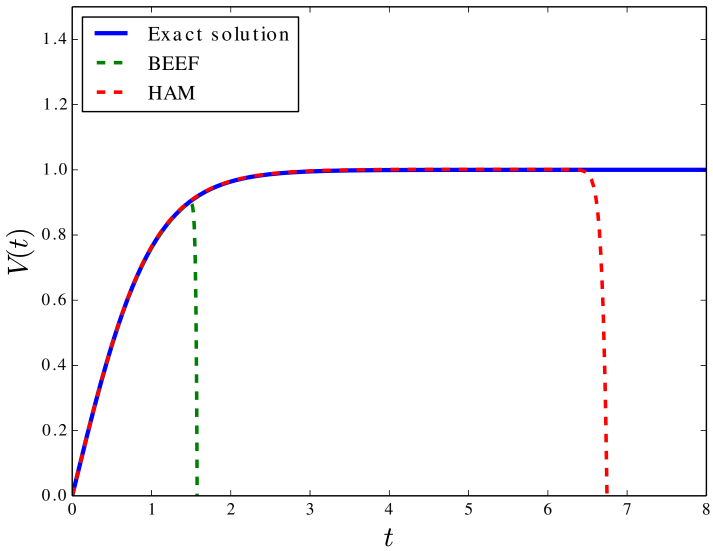

Figure 2: HAM solution taking 100 deformation equations, $\hbar = -0.10$ compared to BEEF taking 100 inner and outer terms with $a = 0$.

analytically that BEEF will converge for values of $|t| < \frac{\pi}{2}$. HAM however has a larger radius of convergence and appears to converge for values $|t| \leq 6.35$.

Clearly both HAM and BEEF solutions will produce accurate representations of the exact solutions over a relatively small domain with HAM converging over a domain about 4 times larger. However one advantage of BEEF is that we can re-centre the series at non-zero values. Through experimentation it was determined that BEEF could be re-centred between the values of $-1.5 \leq a \leq 1.5$ and still produce a convergent solution. Figure 3 shows BEEF centred at $a = 1.5$ and despite being able to provide a accurate representation over a wider positive domain, the HAM method still converges over a larger radius. These results indicate there is evidence to suggest HAM is a more appropriate choice for solving this particular non-linear differential equation.

4.5 Comparing HAM and Padé Solutions

We now compare the HAM solution to the Padé approximation. We go about forming the Padé approximate from the truncated Taylor series formed from BEEF using $a = 0$. Below we briefly demonstrate how to calculate the Padé approximate for this problem.

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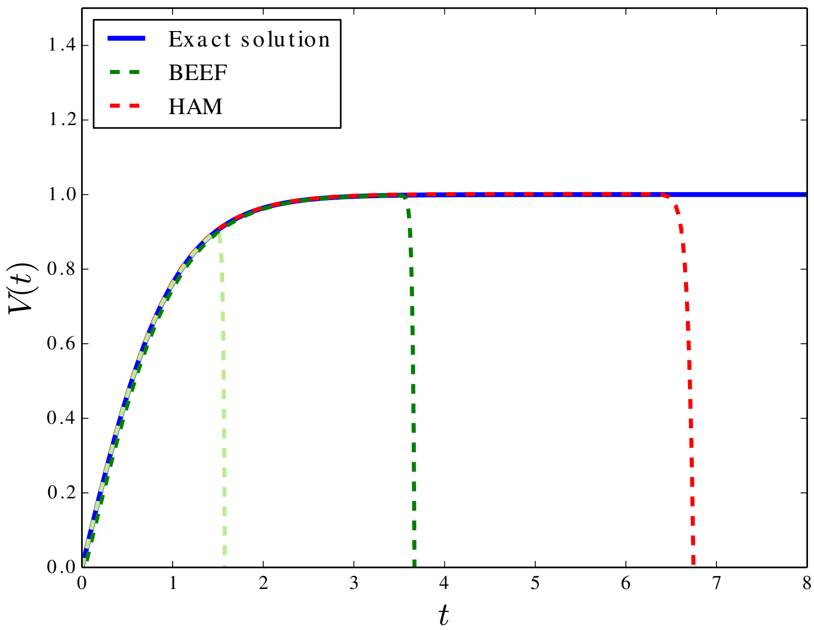

Figure 3: HAM solution taking 100 deformation equations, $\hbar = -0.10$ compared to BEEF taking 100 inner and outer terms with $a = 1.5$. The light green curve is BEEF centred at 0

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$$
\begin{aligned}
V(t) &= R_N(t) \\
t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + ... &= \frac{a_0 + a_1 t + a_2 t^2 + ... + a_n t_n}{1 + b_1 t + b_2 t^2 + ... + b_m t_m}
\end{aligned} \tag{43}
$$

By equating coefficients we can form Padé approximates for the differential equation. For example, we go about computing $R_{[2/2]}$:

$$
t - \frac{t^3}{3} = \frac{a_0 + a_1 t + a_2 t^2}{1 + b_1 t + b_2 t^2} \tag{44}
$$

$$
\begin{aligned}
a_0 &= c_0 = 0 \\
a_1 &= c_1 = 1 \\
a_2 &= b_1 c_1 = b_1
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\frac{-1}{3} + 0 + b_2 &= 0 \\
0 + \frac{-1}{3} b_1 + 0 &= 0
\end{aligned} \tag{46}
$$

From this we get

$$
R_{[2/2]} = \frac{t}{\frac{t^2}{3} + 1} \tag{47}
$$

Similarly we can get further solutions

$$
\begin{aligned}
R_{[3/2]} &= \frac{\frac{2t^3}{3} + t}{\frac{2t^2}{5} + 1} \\
R_{[1/4]} &= \frac{t}{-\frac{2t^4}{90} + \frac{t^2}{3} + 1}
\end{aligned} \tag{48}
$$

etc. ...

Figure 4 shows that when comparing $R_{[30/30]}$ with the HAM solution using the 30th order deformation we see that the Padé approximate matches the exact solution for

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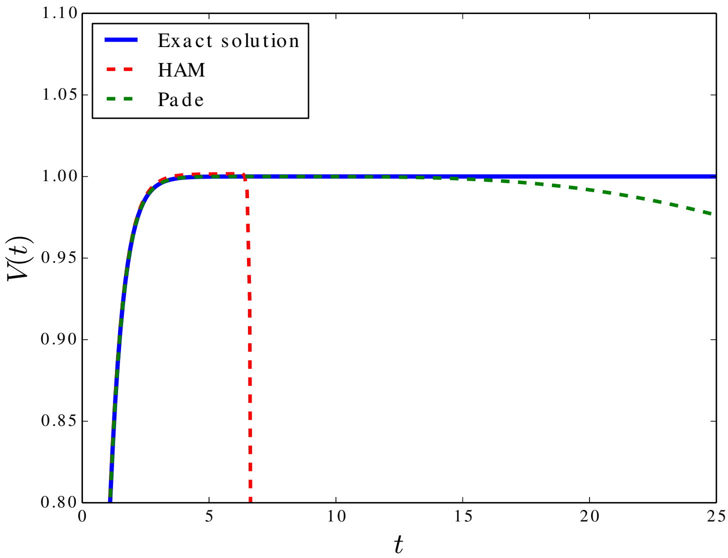

Figure 4: HAM solution using 30th order deformation equation compared to Padé approximate, $R_{[20/20]}$

approximately $t < |12|$. This is almost twice as large as the radius of convergence of HAM suggesting that the Padé approximation is more effective for solving this non-linear differential equation.

A Padé approximate was taken from the HAM polynomial however this did not provide any greater convergence than HAM.

4.6 Equivalence of Methods

We previously noted that when taking the BEEF solution centred at 0 and the HAM solution with $\hbar = -1$ both methods produce the Mclaurin series for $\tanh(t)$. Also based on our choice of base function (polynomial) the HAM solution produced a power series as does the BEEF solution. From this it was hypothesised that there may be a simple relationship between the two methods based on our choices for forming the HAM solution.

We list the mth order deformation equation for HAM using the initial approximation $V_0(t) = t$

$$
\begin{aligned}
V_1(t) &= \frac{1}{3}\hbar t_3 \\
V_2(t) &= \frac{1}{3}\hbar(1-\hbar)t^3 + \frac{2}{15}\hbar^2 t^5 \\
V_3(t) &= \frac{1}{3}\hbar(1+\hbar)^2 t^3 + \frac{2}{15}\hbar^2(1+\hbar)t^5 + \frac{17}{315}\hbar^3 t^7
\end{aligned} \tag{49}
$$

By observation, we can deduce that the coefficient of the cubic term, c_3 using up to the mth order deformation is given by:

$$
c_3 = \frac{\hbar}{3}\left((1+\hbar)^m + m - 1\right) \tag{50}
$$

Also using equation 23 we can define the BEEF coefficient of the cubic term taking m terms of the outer series

$$
c_3 = \sum_{n=3}^{m} \frac{1}{n!}\binom{n}{n-3} f^{(n)}(a)(-a)^{n-3} \tag{51}
$$

Currently no simple relationship between these two coefficients (or higher order) has been found and it is likely that if there is a relationship between the two, it is non-linear or highly complex.

4.7 Initial approximation

An important property of HAM is that any initial approximation which satisfies the initial condition (in this case $V(0) = 0$) can be used. So far we have only used the linear term $V_0(t) = t$ and will now go about comparing different initial approximations. Figure 5 shows that while all four initial approximations provided an accurate representation of the exact solution over a small domain, the optimal choice was the linear term. Note that while $V_0(t) = t, t^2, t^3$ were taken to the 20th deformation, $V_0(t) = \sin(t)$ was only taken to the 7th deformation due to computational restraints.

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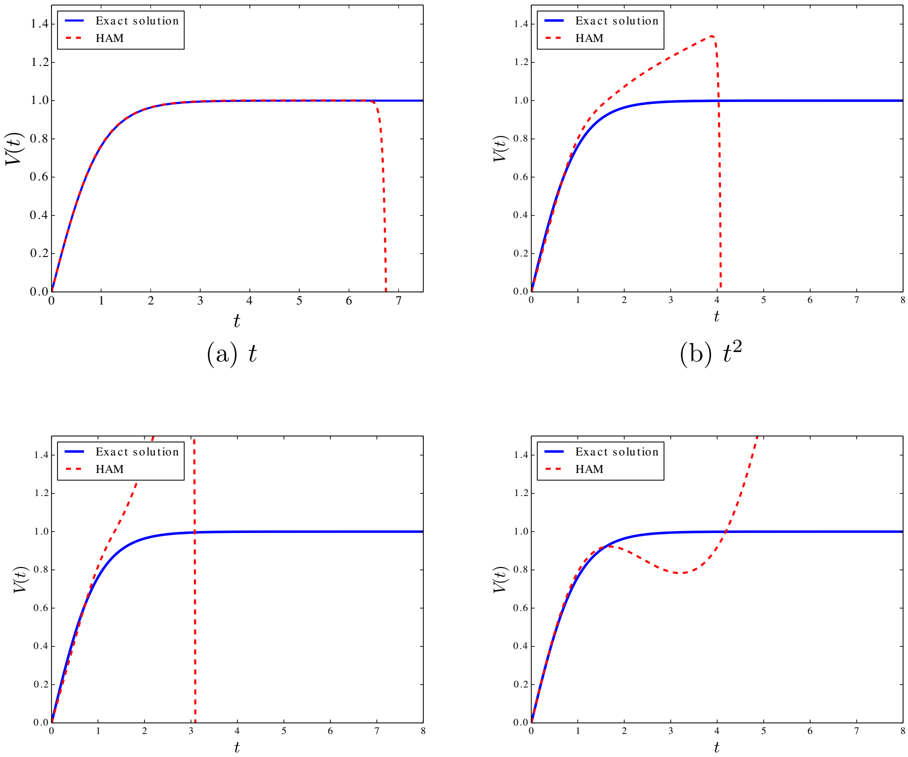

(a) t (b) t^2

(c) t^3 (d) $\sin(t)$, (7th)

Figure 5: Comparison of different initial approximations of HAM using $\hbar = -0.1$ and 20th order deformation with exact solution. $V_0(t) = \sin(t)$ was only taken to 7th order deformation due to computational restraints

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5 Conclusion

In this paper we have have presented three different methods for obtaining analytical solutions to non-linear differential equations. Our focus was to explore the Homotopy Analysis Method, a relatively new technique based on the concept of homotopy and compare it to two well established methods, the Taylor series and Padé approximation. This was done through examination of the non-linear IVP $V'(t) = 1 - V(t)^2$ with $V(0) = 0$ (which has the exact solution $\tanh(t)$) and computing solutions with all three methods. Computation of the HAM method shows that through careful selection of $\hbar$ we were able to obtain a radius of convergence of about 6.35 for a series solution which appears to be centred at the origin.

This is significantly larger than the BEEF's radius of convergence which was $\frac{\pi}{2}$. By taking an approximation of $V(a)$ we were able to re-centre the series between $-1.5 \leq a \leq 1.5$. Despite this, the HAM solution was still able to provide an accurate approximation of the exact solution over a wider domain. However by taking the BEEF polynomial and using it to generate a Padé approximate, we saw this method had the largest radius of convergence of about 12. This suggests that the Padé approximate was the most successful for producing a solution to the non-linear differential equation.

This investigation did reveal several advantages which sets HAM apart from the other methods. Perhaps the most significant is the auxiliary parameter $\hbar$ which as discussed in section 4.3 allows control of the convergence of the series. However several disadvantage were also discovered, notably the method's complexity especially compared to the Taylor series which is typically taught to first year calculus students. This paper has shown that while HAM certainly is capable of producing analytical solutions to differential equations, its position amongst other well established methods may be hard to justify.

6 References

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