The Willmore Energy

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1 Willmore Energy

1.1 Introduction and Definition

The Willmore energy of a (smooth, compact) surface immersed in three dimensional Euclidean space is best thought of as a measure of 'roundness'. For example, the perfectly round sphere (of any radius) is the *global* minimiser of the energy (it has energy 4π as we will see), while a sphere that has been twisted, stretched and folded in on itself will have much higher Willmore energy.

With this intuition, as is customary, we begin with a definition.

Definition. *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a compact orientable surface. The Willmore energy $\mathcal{W}$ of f is given by*

$$\mathcal{W}(f) = \int_\Sigma H^2 dA \tag{1}$$

where H denotes the mean curvature, i.e. mean of the principal curvatures $\frac{\kappa_1+\kappa_2}{2}$.

1.2 Properties

It may not be immediately clear how the above quantity might behave, so it is helpful to enumerate two basic (though perhaps surprising) properties of the Willmore energy.

Theorem. *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a compact orientable surface. The Willmore energy of f is bounded below by 4π with equality obtained if and only if $f(\Sigma)$ is a round sphere.*

Theorem. *The Willmore energy is invariant under conformal (i.e. angle preserving) transformations of $\mathbb{R}^3$.*

The first property, while perhaps not immediately obvious, follows from a straightfoward argument; a succinct presentation can be found in [11]. The second property, conformal invariance of the energy, warrants a little more discussion.

According to a theorem of Liouville [1], any conformal transformation of $\mathbb{R}^3$ is a composition of at most one of each of translations, dilations, reflections and inversions. The invariance under translation and reflection is hardly surprising, as these simply

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correspond to rigid motions of our surface. Dilations again should not require too much imagination - consider doubling the size of our surface, the mean curvature at each point will certainly be less, however we are integrating over a larger area, and after some basic calculation, we can see that these factors do indeed correct each other in the appropriate manner. Invariance under the last (basic) type of conformal transformation, inversion, is much less clear. A proof of this surpising fact is also contained in [11].

2 The Willmore Conjecture

2.1 The Conjecture

In his note on embedded surfaces[10], Willmore conjectured that the minimum Willmore energy for surfaces of genus > 0 was $2\pi^2$. His conjecture was later strengthened to posit that the minimising surface was conformally a smoothly immersed Clifford Torus, leading to the following:

Conjecture (Willmore Conjecture)**.** *Let $f : \Sigma \to \mathbb{R}^3$ be a smooth immersion of a compact orientable surface of genus 1. Then*

$$\mathcal{W}(f) \geq 2\pi^2$$

with equality if and only if $f(\Sigma)$ is conformally the Clifford torus.

2.2 History

Between the time Willmore first stated the conjecture in 1965 and its resolution in 2012, much work was done by a number of people yielding many interim results adding weight to the truth of the conjecture.

Possibly one of the most important papers came from Weiner in 1978[9], establishing the connection between minimal surfaces in $\mathbb{S}^3$ and Willmore surfaces in $\mathbb{R}^3$. Playing into the importance of this result was an earlier construction of minimal surfaces in $\mathbb{S}^3$ of arbitrary genus by Lawson[3].

In 1982, Li and Yau[4] discovered that for a self intersecting immersion f covering a point k times, the Willmore energy of f was bounded below by $4k\pi > 2\pi^2$, allowing

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efforts to be focused on embeddings rather than more general immersions. Furthermore, Langevin and Rosenberg[2] showed that any knotted embedding of a torus in $\mathbb{R}^3$ was bounded below by 8π, hence eliminating the need to consider embeddings in non-standard isotopy groups.

Pinkall[6] in 1985 constructed infinitely many Willmore surfaces that were not conformal to minimal surfaces in $\mathbb{S}^3$.

The existence of a minimising surface in the class of genus 1 surfaces was established by Leon Simon[7] in 1993.

In addition to the above results, the conjecture had been shown to hold for a large number of specific families of torii, giving the conjecture a true *so close, yet so far* flavour.

Finally, in 2013, André Neves and Fernando Codá Marques presented their paper [5] giving resolution to the conjecture.

2.3 Why was it so hard?

So why exactly did the Willmore conjecture take almost 50 years to resolve? As a starting point, we should convince the reader that it is indeed harder than it may first seem. Our first step towards solving the conjecture surely must be to somehow relate the genus of a surface to its Willmore energy. An initial attempt may be a simple application of the Gauss-Bonnet theorem like so:

$$\mathcal{W}(f) = \int_{\Sigma} H^2 dA \geq \int_{\Sigma} K dA = 4\pi(1-g)$$

where g is the genus of the surface. This perhaps verifies our above claims of a global minimiser of genus zero, however for higher genus, it gives very little information, as the lower bound on the right actually *shrinks* as g increases.

A slightly more subtle approach may use the harmonic energy of the Gauss map, $E(G)$, since we have

$$\mathcal{W}(f) = \frac{1}{2}\left(E(G) + \int_{\Sigma} K dA\right).$$

But further calculation shows that

$$\frac{1}{2}\left(E(G)+\int\limits_{\Sigma} KdA\right) \geq \frac{1}{2}\left(\int\limits_{\Sigma} |K|dA+\int\limits_{\Sigma} KdA\right) \geq \int\limits_{\Sigma} KdA$$

and we find ourselves with the same problem. Of course these are quite naive attempts, but their failure points towards a problem inherent in any approach of this type - that the Gauss map may be conformal for spheres, but not for torii [8].

On a deeper level, a main roadblock is that conformal classes of torii are not even closed under particular limits, i.e. we have sequences of torii that do not converge to torii. For example, consider 'shrinking' the inner hole of a torii, we can see that in the limit this sequence will converge to a surface of genus zero, which of course has notably different Willmore properties.

3 Outcomes

During the Summer, I explored the Willmore conjecture and along the way learned much about geometry, knowledge I hope to expand upon and apply in my honours year, which will investigate the proof of Marques and Neves.

The Big Day In was an invaluable experience and helped me develop skills in communicating my work with people in other fields of mathematics and science.

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