AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

National Collaboration in the Mathematical Sciences

Intriguing Sets of Partial Quadrangles

Author:
Supervisor:
MURRAY SMITH DR. JOHN
BAMBERG

Finite geometry is the study of incidence structures, a system of points and lines with some points on some lines according to an incidence relation, such that the number of points and lines is finite. Generalised quadrangles arose from an effort by Tits [5] to give geometric constructions of the finite simple groups of Lie type. Tits' key observation was that the incidence graph of an *n*-gon always has diameter *n* and girth *2n*. So when considering generalised quadrangles, we were concerned with finite incident structures with incidence graph of diameter 4 and girth 8.

In particular, we focused on partial quadrangles, defined by Cameron [2]. Triangle-free strongly regular graphs, of which only seven are known [3], were our source of partial quadrangles, as they satisfy all the axioms of a partial quadrangle. A generalised quadrangle is also a partial quadrangle.

The focus of this research was *intriguing sets,* found as tight sets and ovoids in finite geometry, upper and lower bounds of average degrees in graph theory and correspond to perfect error correcting codes of radius *1* in coding theory. We take a graph with adjacency matrix *A* and a map *f* that sends vertices to real numbers. We are looking for real constants α and β such that

$$Af = \alpha + \beta\chi$$

with χ representing the "all ones" map. If these constants exist then *f* is an *intriguing map*. We were concerned with maps of *1*'s and *0*'s, as this forms a subset relation with the resulting subset an *intriguing set*.

Many of the smaller triangle-free strongly regular graphs, such as the Petersen graph and Clebsch graph have been completely catalogued in terms of intriguing sets [1]. There were gaps in the the largest graphs, such as M_{22} and the *Higman-Sims*, and these would later form the focus of our research.

I started my investigation with *Latin square graphs*. Latin squares are a square array of numbers *1..n* such that no row or column contains duplicate entries. Sudoku puzzles are perhaps the best examples of such Latin squares. A graph is constructed from such a square by considering every entry as a vertex with two entries adjacent if they are in the same row, column or are the same value. This forms a strongly regular graph of parameters $(n^2, 3(n-1), n, 6)$. A *transversal* of a Latin square is a collection of entries *1..n* such that no two entries are in the same row or column or *symbol*, and is an analogue of an ovoid in finite geometry. We proved that a transversal is an intriguing set. Finally, we examined a magical map, analogue of a tight set, in which every vertex is assigned a value equal to the sum of its row and column indices, minus its value. Whilst not being a map of *1*'s and *0*'s, this is intriguing, with $\alpha = n-3$ and $\beta = (n^2 + n)$.

Returning to triangle-free strongly regular graphs, we were able to prove some lemmas that made our investigations easier. By decomposing the adjacency matrix into orthogonal idempotents, we were able to reduce the equation to *n* variables and *n* constraints for a graph of order n, although in practice the rank of these constraints is lower. Furthermore, we were able to narrow down the possible sizes of intriguing sets by proving a lemma on intersections of intriguing sets. We also noted that the complement of an intriguing set is intriguing, so we only needed to consider sets of size less than or equal to *n/2*. Using Gurobi [4] for linear optimisation we were able to find all intriguing sets of the Petersen, Clebsch, Gewirtz and Hoffman-Singleton again.

For larger graphs, though, this was insufficient. The Higman-Sims, for example, is on *100* vertices, and this yielded far too many intriguing sets to calculate in a reasonable time frame. We were able to use *group theory* to break the problem down into smaller chunks. An *automorphism* of a graph is an *isomorphism* from the graph to itself. The orbit of an intriguing set under the group of automorphisms can all be considered equivalent; we only need to find one, in order to find the others.

All of the graphs we considered were *vertex-transitive*, meaning that for any two vertices there existed an automorphism mapping one to the other. Hence, we assumed that an arbitrary vertex would be in a solution, reducing the total number of solutions we needed to find. Furthermore, since intriguing sets under automorphism are considered equivalent, we were able to reduce the number of solutions even further by examining the orbits of the stabiliser of that arbitrary vertex. All of our graphs were of *permutation rank 3* so, excluding the trivial orbit containing the vertex we stabilised, there were only *2* orbits. We could then split the problem into two cases; one involving an arbitrary fixed element from the first vertex and another involving an arbitrary fixed element from the second. The cases could then be solved and combined. This process, which we called *symmetry breaking*, could be continued depending upon how many vertices we wanted to assume were in a solution.

This symmetry breaking simplified the computation, allowing us to gather some results. We found that all intriguing sets of size *50* in the Higman-Sims were isomorphic to the Hoffman-Singleton graph, and that all sets of size *10* were isomorphic to either a circulant with joining set *{2,3,7,8}*, or $K_{5,5}$ minus a matching. M_{22}, however, was different, in that we found an intriguing set of size *33* with a trivial stabiliser, leading to *887040* isomorphic sets. We then reduced these sets and their complements to unions of intriguing sets of size *11*. This partitions the vertices into intriguing sets, and would suggest that there are 2^7-2 nontrivial intriguing sets for each such partition. Furthermore, the stabiliser of this partition is also trivial, so there are plenty of ways to partition the vertices. This suggests that there are far too many to be interested in singular examples.

Further work would revolve around *intriguing partitions*; intriguing sets that can be reduced to the union of at least *3* non-trivial intriguing sets. The results of M_{22} clearly position this as an area of interest.

The AMSI Vacation Research Scholarship has been immensely valuable to me, allowing me to experience research in mathematics and to learn about recent advances in finite geometry. Importantly, I was able to meet and network with similarly minded mathematics students from around the nation. Thanks are due for my supervisor Dr. John Bamberg, AMSI, CSIRO and the University of Western Australia.

References

[1] Bamberg, John; De Clerck, Frank; Durante, Nicola. Intriguing sets in partial quadrangles. *J. Combin. Des.* 19 (2011), no. 3, 217–245
[2] P. Cameron. Partial quadrangles. *Q. J. Math*. Oxford, 25:1–13, 1974.
[3] C. Godsil. Problems in algebraic combinatorics. *Journal of Electronic Combinatorics*, 2,
1995.
[4] Gurobi Optimization, Inc. Gurobi Optimizer Version 3.0, 2010.
[5] J. Tits. Sur la trialité et certains groups qui s'en déduisent. *Publ. Math. IHES Paris*, 2:16–60, 1959.

Postal Address: Building 161
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 8344 6324
Web: www.amsi.org.au