Finite Simple Groups, Trees and Shabat Polynomials

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My vacation research project was supervised by Associate Professor Finnur Larusson at the University of Adelaide.

The goal of the project was to investigate the correspondence between finite simple groups, a type of graphs known as dessins d'enfants and Belyi maps. Then for some cases when the dessin d'enfant is a tree the goal was to compute the corresponding Belyi map, which is a Shabat polynomial.

The finite simple groups I studied were the Mathieu groups, some of the smallest finite simple groups. In particular a permutation representation of each Mathieu group was needed in order to draw the corresponding dessin d'enfant. I began by finding these permutation representations from the ATLAS of finite simple groups [1].

While looking into the origin of the permutation representations I came across some nice characterisations of the Mathieu groups that arise from the classification of finite simple groups. In particular, other than the symmetric and alternating groups S_n and A_n for $n \geq 4$, M_{23} and M_{11} are the only groups that act 4-transitively and M_{24} and M_{12} are the only groups that act 5-transitively on their respective sets (i.e. M_{11} acts on a set of 11 elements).

With the permutation representations in hand it was possible to draw up the corresponding dessins d'enfants. It turns out that only the dessins d'enfants corresponding to M_{11} and M_{23} are *trees*; a finite connected graph with no loops.

Having drawn these two trees the next thing I did was write a program in the computer algebra system Maple that attempts to solve for the Shabat polynomial corresponding to a tree. This involves solving a system of nonlinear equations to find the coefficients of the polynomial.

The degree of the polynomial is equal to the number of vertices of the tree, which in turn is equal to the size of the set the permutation representation acts on. For M_{11} the Shabat polynomial has degree 11, though we are allowed to make an arbitrary choice for one of the coefficients as the polynomial is only unique up to affine transformation.

Algebraically solving a system of 11 nonlinear equations is not a trivial task, even for a computer. To make matters worse, when constructing the equations it is not possible to ensure a unique solution. Instead a solution exists for each different tree with the same valencies of the vertices. To find the desired solution it is necessary to compute all the others as well.

Much of my time consisted of trying to make the above calculations computationally easier. There are a number of ways of doing this, such as careful consideration of what should be chosen and what should be the unknowns.

The main result of the project was successfully computing the Shabat polynomial corresponding to M_{11}. This polynomial is unique up to affine transformation, and represents all 7920 elements of M_{11}.

The project has been a great opportunity to learn some maths in an interesting area and has also been a great experience in preparing me for further study. In particular, the experiences of working with a supervisor and writing a mathematics based report were new for me and will be extremely handy in the future.

References:

[1] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott. ATLAS of Finite Group Representations. http://brauer.maths.qmul.ac.uk/Atlas/v3/