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Investigation of manipulation of Methods of Difference Construction for Mutually Nearly Orthogonal Latin Squares

Helen Hunt

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I here aim to investigate methods of modifying existing systems of cyclic mutually nearly orthogonal latin squares to generate further sets of mutually nearly orthogonal latin squares.

1 Preliminaries

1.1 Latin Squares

A *latin square* of order n is an $n \times n$ array wherein every item of an n-set, N, appears precisely once in each row and each column. Here, we use $N = \{0, \dots, n-1\}$.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

1.2 Mutually Orthogonal

Given two latin squares of order n on the set N, $L = [l_{ij}]$ and $M = [m_{ij}]$, we define the superimposition of L onto M as the $n \times n$ array $A = [(l_{ij}, m_{ij})_{ij}]$ where i, j are in N. L and M are said to be *orthogonal* if every ordered pair (α, β) for $\alpha, \beta \in N$ appears precisely once in A. A set of latin squares are said to be *mutually orthogonal* if they are pairwise orthogonal, denoted MOLS(n).

Mutually orthogonal latin squares are a tool of experimentation allowing for heterogeneity to be eliminated in two directions and different interventions investigated (Raghavarao, Shrikhande & Shrikhande 2002). For instance, Tippett (1924) showed a complete set of mutually orthogonal latin squares can be used to reduce an experimental design of 5^6 runs to an experimental design of 5^2 runs, whilst ensuring that all interventions are tested with all others in a pairwise manner.

The maximum size of a set of MOLS(n) may be restrictive to their utility. For instance, the maximum size of a set of MOLS(6) is one. More generally, the maximum size of a set of MOLS($2m$) for a positive integer m may be small (Pasles 2003). In order to generalise the experimental utility of mutually orthogonal latin squares, Williams (1949) defined a residual design – mutually nearly orthogonal latin squares.

1.3 Mutually Nearly Orthogonal

Two latin squares of even order $2m$ on the set $\{0, \ldots, 2m-1\}$, L and M, are said to be *nearly orthogonal* if the superimposition of L onto M yields an array A containing each ordered pair (α, β) for $\alpha, \beta \in N, \alpha \neq \beta$ at least once and the ordered pair $(\alpha, \alpha + m)$ precisely twice. Similarly, a set of latin squares which are pairwise nearly orthogonal are called mutually nearly orthogonal, denoted MNOLS($2m$).

1.4 Cyclically Generated

A $1 \times n$ vector can be used to denote a latin square of order n in the following way.

Given a column vector $C = (a_0, \ldots, a_{n-1})$, some permutation of the elements of $\{0, \ldots, n-1\}$, C represents an array wherein each cell, l_{ij} in the array is then given by $l_{ij} = a_{i-1} + j - 1 \pmod n$.
That is,

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c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

$$[l_{ij}] = \begin{array}{|c|c|c|c|c|} \hline a_0 & a_0+1 & \dots & a_0+n-2 & a_0+n-1 \\ \hline a_1 & a_1+1 & \dots & a_1+n-2 & a_1+n-1 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{n-2} & a_{n-2}+1 & \dots & a_{n-2}+n-2 & a_{n-2}+n-1 \\ \hline a_{n-1} & a_{n-1}+1 & \dots & a_{n-1}+n-2 & a_{n-1}+n-1 \\ \hline \end{array}$$

where the entry in each cell is computed (mod n). Then, since each row and each column contains every element of $\{0, \dots, n-1\}$ precisely once, $[l_{i,j}]$ is a latin square. We take $C = (a_0, \dots, a_{n-1})$ to be a representation of the latin square generated in this manner.

2 Method of Differences

Raghavarao, Shrikhande and Shrikhande defined and investigated the Method of Differences construction in their investigation of MNOLS, demonstrating this construction to be highly useful for the generation and representation of cyclic MNOLS. Their theorem (2002) is as follows.

Theorem. *Let there exist t column vectors $(a_{0i}, a_{1i}, \dots, a_{2m-1i})'$ for $i = 1, \dots, t$ where each column vector is a permutation of the elements of the cyclic group $\mathbb{Z}_{2m} = \{0, \dots, 2m-1\}$. Furthermore, suppose for every $i \neq j$, $i, j = 1, \dots t$, among the $2m$ differences $a_{0i} - a_{0j}, a_{1i} - a_{1j}, \dots, a_{2m-1i} - a_{2m-1j}$ (mod $2m$), m occurs twice and all other non-zero elements of $\mathbb{Z}_{2m}$ occur once. Then by taking each of the t columns as the first column of t latin squares and developing them in $\mathbb{Z}_{2m}$, we get t MNOLS.*

2.1 Methods of Difference Property

It is our aim to generate further sets of MNOLS from existing such sets. To this end we say a set of vectors $\{C_1, C_2, \dots, C_t\}$ has the property P_1 if they represent a set of MNOLS constructible using the above method of differences. That is, if, for each $C_i = (a_{0i}, a_{1i}, \dots, a_{n-1i})$ and for every $i \neq j$, $i, j = 1, \dots, t$,among the $2m$ differences $a_{0i} - a_{0j}, a_{1i} - a_{1j}, \dots, a_{2m-1i} - a_{2m-1j}$ (mod $2m$), m occurs twice and all other non-zero elements of $\mathbb{Z}_{2m}$ occur once, then the set $\{C_1, C_2, \dots, C_t\}$ is said to possess P_1. Hence by Raghavarao, Shrikhande and Shrikhande's theorem, the array represents a set of t MNOLS($2m$).

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

2.2 Difference Vectors

Furthermore, we utilise stepwise difference vectors. That is, for $\{C_1, C_2, \ldots, C_t\}$ we define t $1 \times 2m$ vectors Δ_i, $i = 1, \ldots, t$ where

$$\Delta_i = (\delta_{0i}, \delta_{1i}, \ldots, \delta_{2m-2i}, \delta_{2m-1i}) \text{ and}$$
$$\delta_{\beta i} \equiv a_{\beta i+1} - a_{\beta i} \pmod{2m}.$$

Herein, any subscript $i \pm 1$ and $j \pm 1$ is always taken to mean $i \pm 1 \pmod{t}$ and $j \pm 1 \pmod{t}$ respectively. We note that $\{C_1, C_2, \ldots, C_t\}$ has property P_1 if $\forall i \in \{1, \ldots, t\}$, Δ_i is some permutation of the elements of the multiset $\{1, 2, \ldots, m-1, m, m, m+1, \ldots, 2m-2, 2m-1\}$.

We note that if $t = 3$, the set $\{\Delta_i | 1 \leq i \leq 3\}$ represents all possible comparisons of $\{C_1, C_2, C_3\}$. Hence, for a set $\{C_1, C_2, C_3\}$ merely checking that for each $i = 1, 2, 3$, Δ_i contains each non-zero element of $\mathbb{Z}_{2m}$ at least once and m precisely twice is sufficient to show that $\{C_1, C_2, C_3\}$ has property P_1 and represents a set of 3 MNOLS($2m$).

A further property of our construction of $\{\Delta_i | 1 \leq i \leq t\}$ is that $\forall \beta \in \{0, \ldots, 2m-1\}$

$$\sum_{i=1}^{t} \delta_{\beta,i} = a_{\beta 2} - a_{\beta 1} + a_{\beta 3} - a_{\beta 2} + \cdots + a_{\beta t-1} - a_{\beta t} + a_{\beta 1} - a_{\beta t}$$
$$\equiv 0 \pmod{2m}. \qquad (1)$$

3 Permuting entries of a Set with the Property P_1

We investigate when modifying a set with property P_1 by permuting the entries of one C_i of the set with some $2m - x$ fixed points yields a set also possessing property P_1.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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3.1 $x = 1$

Claim: Assume $\{C_1, C_2, \dots, C_t\}$ is a set of t vectors with P_1. It is not possible to, by modifying precisely one C_j, $j \in \{1, \dots, t\}$, by permuting its entries with $2m-1$ fixed points, yield another set with P_1.

Proof.

Suppose that we have two sets of t vectors both possessing P_1; $\{C_1, C_2, \dots, C_t\}$ and $\{C'_1, C'_2, \dots, C'_t\}$. Let it be the case that for all $i \in \{1, \dots, t\} \backslash \{j\}$, $C_i = C'_i$ and for all $\beta \in \{0, \dots, 2m-1\} \backslash \{x_1\}$, $a_{\beta j} = a'_{\beta j}$ but $a_{x_1 j} \neq a'_{x_1 j}$. Since C_j is a permutation of length $2m$ of the elements of $\mathbb{Z}_{2m}$, every element of the set $\{0, \dots, 2m-1\}$ occurs once in C_j. But, if $\forall \beta \in \{0, \dots, 2m-1\} \backslash \{x_1\}$, $a_{\beta j} = a'_{\beta j}$ then $a_{x_1 j} = a'_{x_1 j}$. This is a contradiction. Hence, it is not possible to modify just one entry of one C_j, $j \in \{1, \dots, t\}$, of a set $\{C_1, C_2, \dots, C_t\}$ possessing P_1, creating a set also with P_1.

3.2 $x = 2$

Claim: Assume $\{C_1, C_2, \dots, C_t\}$ is a set of t vectors with property P_1. It is not possible to, by modifying precisely one C_j, $j \in \{1, \dots, t\}$, by permuting its entries with $2m-2$ fixed points, yield another set with P_1.

Proof.

Let there be two distinct sets of size t both possessing property P_1; $\{C_1, C_2, \dots, C_t\}$ and $\{C'_1, C'_2, \dots, C'_t\}$. Let it be the case that $\forall i \in \{1, \dots, t\} \backslash \{j\}$, $C_i = C'_i$ and for C_j, for all $\beta \in \{0, \dots, 2m-1\} \backslash \{x_1, x_2\}$, $a_{\beta j} = a'_{\beta j}$ but $a_{x_1 j} \neq a'_{x_1 j}$ and $a_{x_2 j} \neq a'_{x_2 j}$.

Since C_j is a permutation of $\mathbb{Z}_{2m}$, each element of $\{0, \dots, 2m-1\}$ occurs precisely once in C_j. Hence, $\{a_{x_1 j}, a_{x_2 j}\} = \{a'_{x_1 j}, a'_{x_2 j}\}$. Thus $a_{x_1 j} = a'_{x_2 j}$ and $a_{x_2 j} = a'_{x_1 j}$.

In addition, we have the requirement that in our stepwise distance vectors are a permutation the multiset $\{1, 2, \dots, m-1, m, m, m+1, \dots, 2m-2, 2m-1\}$. All $\delta_{\beta i}$ are unchanged for $\beta \in \{0, \dots, 2m-1\} \backslash \{x_1, x_2\}$ and $\delta_{x_1 i} = \delta'_{x_1 i}$ for all $i \in \{1, \dots, t\} \backslash \{j, j-1\}$. Furthermore, we require that

$$\{\delta_{x_1 j}, \delta_{x_2 j}\} = \{\delta'_{x_1 j}, \delta'_{x_2 j}\}$$

and

$$\{\delta_{x_1 j-1}, \delta_{x_2 j-1}\} = \{\delta'_{x_1 j}, \delta'_{x_2 j}\}.$$

This gives us two cases; that $\delta_{x_1 j} = \delta'_{x_1 j}$ and $\delta_{x_2 j} = \delta'_{x_2 j}$ or $\delta_{x_1 j} = \delta'_{x_2 j}$ and $\delta_{x_2 j} = \delta'_{x_1 j}$.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

1. In the first case, we have that since

$$\begin{aligned} \delta_{x_1j} &= \delta'_{x_1j} && \text{then} \\ a_{x_1j+1} - a_{x_1j} &= a'_{x_1j+1} - a'_{x_1j} \\ \text{since we have that} \\ a_{x_1j+1} &= a'_{x_1j+1} && \text{clearly} \\ a_{x_1j} &= a'_{x_1j}. \end{aligned}$$

This is a contradiction since a_{x_1j} cannot equal a'_{x_1j}.

2. In the second case, we see that,

$$\begin{aligned} \sum_{i=1}^{t} \delta_{\beta i} &\equiv 0 \pmod{2m} \text{ by equation (1)} \\ \text{so } \sum_{i=1}^{t} \delta_{\beta i} &\equiv \sum_{i=1}^{t} \delta'_{\beta i} \pmod{2m}. \\ \text{Since } \delta_{x_1i} &= \delta'_{x_1i} \ \forall i \in \{1,\ldots,t\}\backslash\{j, j-1\}, \text{ we must have that} \\ \delta_{x_1j} + \delta_{x_1j-1} &\equiv \delta'_{x_1j} + \delta'_{x_1j-1} \pmod{2m} \\ \text{but } \delta_{x_1j} &= \delta'_{x_2j} \\ \text{so } \delta'_{x_2j} + \delta_{x_1j-1} &\equiv \delta'_{x_1j} + \delta'_{x_1j-1} \pmod{2m}. \text{ In which case,} \\ a'_{x_2j+1} - a'_{x_2j} + a_{x_1j} - a_{x_1j-1} &\equiv a'_{x_1j+1} - a'_{x_1j} + a'_{x_1j} - a'_{x_1j-1} \pmod{2m}. \\ \text{However, } a_{x_1j-1} &= a'_{x_1j-1} \text{ and } a'_{x_2j} = a_{x_1j}. \\ \text{Thus } a'_{x_2j+1} &= a'_{x_1j+1}. \end{aligned}$$

This is a contradiction since $\forall i \in \{1,\ldots,t\}\backslash\{j\}$, $C_i = C'_i$. Therefore, it is not possible to modify a set $\{C_1,\ldots,C_t\}$ with property P_1 by permuting the entries of one C_j, $j \in \{1,\ldots,t\}$, with $2m-2$ fixed points, create a set with the same property.

3.3 $x = 3$

Claim: Assume $\{C_1, C_2, C_3\}$ is a set of 3 vectors with property P_1. It is not possible to, by modifying precisely one C_j, $j \in \{1,\ldots,t\}$, by permuting its entries with $2m-3$ fixed points, yield another set with P_1.

Proof.

Let the set $\{C_1, C_2, C_3\}$ be a set of 3 column vectors where $C_i = (a_{0i}, a_{1i}, \ldots, a_{2m-1i})'$ for $1 \leq i \leq t$ and each C_i is a permutation of the elements of the cyclic group $\mathbb{Z}_{2m}$. Let it be the case that property P_1 holds for $\{C_1, C_2, C_3\}$.

Consider the set $\{C_1', C_2', C_3'\}$ where $\forall i \in \{1,2,3\}\backslash\{j\}$ $C_i = C_i'$ and for all $\alpha \in \mathbb{Z}_{2m}\backslash\{x_1, x_2, x_3\}$, where x_1, x_2 and x_3 are distinct, $a'_{\alpha j} = a_{\alpha j}$ but $a'_{x_n j} \neq a_{x_n j}$ for $n = 1, 2, 3$. Suppose that $\{C_1', C_2', C_3'\}$ also has property P_1.

As $\{C_1', C_2', C_3'\}$ has property P_1 then

$$\begin{aligned}\{a'_{x_1 j}, a'_{x_2 j}, a'_{x_3 j}\} &= \{a_{x_1 j}, a_{x_2 j}, a_{x_3 j}\},\\ \{\delta'_{x_1 j}, \delta'_{x_2 j}, \delta'_{x_3 j}\} &= \{\delta_{x_1 j}, \delta_{x_2 j}, \delta_{x_3 j}\} \text{ and}\\ \{\delta'_{x_1 j-1}, \delta'_{x_2 j-1}, \delta'_{x_3 j-1}\} &= \{\delta_{x_1 j-1}, \delta_{x_2 j-1}, \delta_{x_3 j-1}\}.\end{aligned}$$

Without loss of generality, we label our columns such that $a'_{x_1 j} = a_{x_2 j}$. It is clearly consequent that $a'_{x_2 j} = a_{x_3 j}$ and $a'_{x_3 j} = a_{x_1 j}$.

Furthermore, clearly $\delta_{x_n j+1} = \delta'_{x_n j+1}$ for $n = 1, 2, 3$ since $\delta_{x_n j+1} = a_{x_n j-1} - a_{x_n j+1}$.

We then have two cases; 1. $\delta'_{x_3 j} = \delta_{x_1 j}$ or 2. $\delta'_{x_3 j} = \delta_{x_2 j}$

1. In the first case, then $a_{x_3 j+1} - a_{x_1 j} = a_{x_1 j+1} - a_{x_1 j}$ from definition of δ.

   $\therefore a_{x_3 j+1} = a_{x_1 j+1}$ which is a contradiction.

2. In the second case,

$$\begin{aligned}\delta'_{x_3 j} &= \delta_{x_2 j} \text{ which implies that}\\ \delta'_{x_1 j} &= \delta_{x_3 j} \text{ and}\\ \delta'_{x_2 j} &= \delta_{x_1 j}\end{aligned}$$

   Furthermore, let $\delta'_{x_1 j-1} = \delta_{x_s j-1}$ where $s, t \in \{2, 3\}$, $s \neq t$ and note that if

$$\begin{aligned}\delta'_{x_s j-1} &= \delta_{x_1 j-1} \text{ then}\\ \delta'_{x_t j-1} &= \delta_{x_t j-1}. \text{ This would give that}\\ a'_{x_t j} - a'_{x_t j-1} &= a_{x_t j} - a_{x_t j-1} \text{ which implies that}\\ a'_{x_t j} &= a_{x_t j}\end{aligned}$$

   which is a contraction. Hence, $\delta'_{x_s j-1} = \delta_{x_t j-1}$ and $\delta'_{x_t j-1} = \delta_{x_s j-1}$.

   The following two cases are possible; (a) $s = 2$ and $t = 3$ or (b) $s = 3$ and $t = 2$

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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(a) $s = 2$, $t = 3$

$$\begin{aligned} \text{Then } \delta'_{x_3 j-1} &= \delta_{x_1 j-1} \\ \text{implies that } a'_{x_3 j} - a_{x_3 j-1} &\equiv a_{x_1 j} - a_{x_1 j-1} \pmod{2m} \\ \text{so } a_{x_1 j} - a_{x_3 j-1} &\equiv a_{x_1 j} - a_{x_1 j-1} \pmod{2m} \\ \text{gives that } a_{x_3 j-1} &\equiv a_{x_1 j-1} \pmod{2m} \text{ ※} \end{aligned}$$

(b) $s = 3$, $t = 2$

This gives that since

$$\begin{aligned} -\delta_{x_1 j+1} &\equiv \delta_{x_1 j} + \delta_{x_1 j-1} \pmod{2m} \\ &\equiv \delta'_{x_1 j} + \delta'_{x_1 j-1} \pmod{2m} \\ &\equiv \delta_{x_3 j} + \delta_{x_3 j-1} \pmod{2m} \\ &\equiv -\delta_{x_3 j+1} \pmod{2m} \\ &\equiv \delta'_{x_3 j} + \delta'_{x_3 j-1} \pmod{2m} \\ &\equiv \delta_{x_2 j} + \delta_{x_2 j-1} \pmod{2m} \\ &\equiv -\delta_{x_2 j+1} \pmod{2m} \end{aligned}$$

we have $-\delta_{x_1 j+1} \equiv -\delta_{x_2 j+1} \equiv -\delta_{x_2 j+1} \pmod{2m}$, clearly a contradiction.

Hence, it is not possible to, by modifying 3 entries of one element of a set possessing P_1, create another set with the same property.

3.4 $x = 4$

Let there exist two sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ where each C_i, C'_i for $i \in \{1, 2, 3\}$ is a vector of length $2m$, some permutation of the elements $\mathbb{Z}_{2m}$, such that the P_1 holds for both sets. Suppose that $C_i = C'_i\ \forall i \in \{1, 2, 3\} \backslash \{j\}$ and C'_j is a permutation of the elements of C_j with exactly $2m - 4$ fixed points.

For example, consider the sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ where

C_1	0	1	2	3	4	5	6	7	8	9
C_2 =	1	0	4	8	7	9	2	5	3	6
C_3	5	7	**0**	6	**9**	**4**	**3**	8	2	1

and

$$\begin{array}{c} C'_1 \\ \hline C'_2 \\ \hline C'_3 \end{array} = \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 4 & 8 & 7 & 9 & 2 & 5 & 3 & 6 \\ \hline 5 & 7 & \mathbf{9} & 6 & \mathbf{3} & \mathbf{0} & \mathbf{4} & 8 & 2 & 1 \end{array}.$$

The sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ both possess property P_1 and note that in this example with $2m = 10$, $x = 4$ is possible.

The restrictions upon the case where $x = 4$, however are strong.

3.4.1 Restrictions upon $x = 4$

Assume $\{C_1, C_2, C_3\}$, $\{C'_1, C'_2, C'_3\}$ are sets of vectors with each element of each set a permutation of the elements of $\mathbb{Z}_{2m}$ and let both sets possess property P_1. Let it be the case that $\forall i \in \{1,2,3\} \backslash \{j\}$, $C_i = C'_i$ and C'_j is a permutation of the elements of C_j with exactly $2m - 4$ fixed points. We label the set of positions at which C_j deranged $\{x_1, x_2, x_3, x_4\}$.

Without loss of generality, we label these such that a_{x_1} and a_{x_2j} such that $a_{x_2j} = a'_{x_1j}$. We then are left with two choices; $a'_{x_2j} = a_{x_1j}$ or $a'_{x_2j} \neq a_{x_1j}$.

1. If it were the case that $a'_{x_2j} = a_{x_1j}$ then we would have that $a'_{x_3j} = a_{x_4j}$ and $a'_{x_4j} = a_{x_3j}$. We then have one of the following subcases: $\delta'_{x_1j} = \delta_{x_1j}$, $\delta'_{x_1j} = \delta_{x_2j}$, $\delta'_{x_1j} = \delta_{x_3j}$ or $\delta'_{x_1j} = \delta_{x_4j}$.

   - $\delta'_{x_1j} = \delta_{x_1j} \;\Rightarrow\; a_{x_1j+1} - a'_{x_1j} = a_{x_1j+1} - a_{x_1j} \Rightarrow a_{x_2j} = a_{x_1j}$ $\divideontimes$
   - Similarly, $\delta'_{x_1j} = \delta_{x_2j} \;\Rightarrow\; a_{x_1j+1} - a'_{x_1j} = a_{x_2j+1} - a_{x_2j}$
     $\Rightarrow a_{x_1j+1} = a_{x_2j+1}$ $\divideontimes$
   - Since our labelling is arbitrary and x_3 and x_4 are thus far interchangeable without loss of generality we label x_3 such that $\delta'_{x_1j} = \delta_{x_3j}$.

   We then have that, for the same reasons, $\delta'_{x_2j} = \delta_{x_4j}$.

   Now, considering δ'_{x_1j-1}, we have the subcases that $\delta'_{x_1j-1} = \delta_{x_1j-1}$, $\delta'_{x_1j-1} = \delta_{x_2j-1}$, $\delta'_{x_1j-1} = \delta_{x_3j-1}$ or $\delta'_{x_1j-1} = \delta_{x_4j-1}$

   - $\delta'_{x_1j-1} = \delta_{x_1j-1} \;\Rightarrow\; a_{x_2j} = a_{x_1j}$ $\divideontimes$

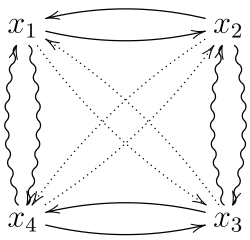
- $\delta'_{x_1 j-1} = \delta_{x_2 j-1} \;\Rightarrow\; a_{x_1 j-1} = a_{x_2 j-1}$ ※

- $\delta'_{x_1 j-1} = \delta_{x_3 j-1}$
  $\Rightarrow\; \delta'_{x_1 j-1} + \delta'_{x_1 j} \equiv \delta_{x_3 j-1} + \delta_{x_3 j} \pmod{2m}$
  $\Rightarrow \delta_{x_1 j+1} \equiv \delta_{x_3 j+1} \equiv m \pmod{2m}$ Furthermore, it must also be the case that $\delta'_{x_2 j} = \delta_{x_4 j}$ and $\delta'_{x_2 j-1} = \delta_{x_4 j-1}$. Hence
  $\delta'_{x_2 j-1} + \delta'_{x_2 j} \equiv \delta_{x_4 j-1} + \delta_{x_4 j} \pmod{2m}$
  $\Rightarrow \delta_{x_2 j+1} \equiv \delta_{x_4 j+1} \equiv m$.
  But then we have 4 occurrences of m in the multiset Δ_{j+1} ※

- $\therefore\; \delta'_{x_1 j-1} = \delta_{x_4 j-1}$ and $\delta'_{x_2 j-1} = \delta_{x_3 j-1}$

We then have two possibilities:

(a) $\delta'_{x_3 j} = \delta_{x_1 j}$. In this case $\delta'_{x_4 j} = \delta_{x_2 j}$, $\delta'_{x_3 j-1} = \delta_{x_2 j-1}$ and $\delta'_{x_4 j-1} = \delta_{x_1 j-1}$. This can be shown graphically by the following;



where a black vector represents the equivalence of $a'_{x_\alpha j}$ to $a_{x_\tau j}$, a dotty vector the equivalence of $\delta'_{x_\alpha j}$ to $\delta_{x_\beta j}$ and a squiggly vector the equivalence of $\delta'_{x_\alpha j-1}$ to $\delta_{x_\gamma j-1}$. In a manner, each vector represents the mapping $f : \mathbb{Z}_{2m} \to \mathbb{Z}_{2m}$ where $f(a) = a'$ and $f(\delta) = \delta'$.

To further indulge this rabbit hole, I note that the system of equations given by this eventuality give that since

$$\delta_{x_1 j} \equiv a_{x_1 j+1} - a_{x_1 j} \quad \equiv a'_{x_1 j+1} - a'_{x_1 j} \equiv a_{x_3 j+1} - a_{x_4 j} \pmod{2m} \qquad (2)$$
$$\delta_{x_2 j} \equiv a_{x_2 j+1} - a_{x_2 j} \quad \equiv a'_{x_2 j+1} - a'_{x_2 j} \equiv a_{x_4 j+1} - a_{x_3 j} \pmod{2m} \qquad (3)$$

addition of equations (2) and (3) give that

$$a_{x_1 j+1} - a_{x_1 j} + a_{x_2 j+1} - a_{x_2 j} \equiv a_{x_3 j+1} - a_{x_3 j} + a_{x_4 j+1} - a_{x_4 j} \pmod{2m}$$
$$\Rightarrow \delta_{x_1 j} + \delta_{x_2 j} \equiv \delta_{x_3 j} + \delta_{x_4 j} \pmod{2m}$$

Furthermore,

$$\delta_{x_1j-1} \equiv a_{x_1j} - a_{x_1j-1} \quad \equiv a'_{x_1j} - a'_{x_1j-1} \equiv a_{x_3j} - a_{x_4j-1} \pmod{2m} \quad (4)$$
$$\delta_{x_2j-1} \equiv a_{x_2j} - a_{x_2j-1} \quad \equiv a'_{x_2j} - a'_{x_2j-1} \equiv a_{x_4j} - a_{x_3j-1} \pmod{2m} \quad (5)$$

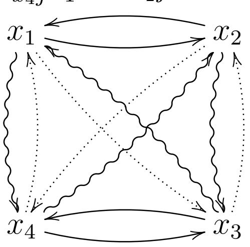
addition of equations (4) and (5) give that

$$a_{x_1j} - a_{x_1j-1} + a_{x_2j} - a_{x_2j-1} \equiv a_{x_3j} - a_{x_3j-1} + a_{x_4j} - a_{x_4j-1} \pmod{2m}$$
$$\Rightarrow \delta_{x_1j-1} + \delta_{x_2j-1} \equiv \delta_{x_3j-1} + \delta_{x_4j-1} \pmod{2m}$$

It follows that

$$\delta_{x_1j+1} + \delta_{x_2j+1} \equiv \delta_{x_3j+1} + \delta_{x_4j+1} \pmod{2m}$$

(b) $\delta'_{x_3j} = \delta_{x_2j}$. In this case, $\delta'_{x_4j} = \delta_{x_1j}$, $\delta'_{x_3j-1} = \delta_{x_1j-1}$ and $\delta'_{x_4j-1} = \delta_{x_2j-1}$. This can be shown graphically by the following;



where a black vector represents the equivalence of $a'_{x_\alpha j}$ to $a_{x_\tau j}$, a dotty vector the equivalence of $\delta'_{x_\alpha j}$ to $\delta_{x_\beta j}$ and a squiggly vector the equivalence of $\delta'_{x_\alpha j-1}$ to $\delta_{x_\gamma j-1}$. In a manner, each vector represents the mapping $f : \mathbb{Z}_{2m} \rightarrow \mathbb{Z}_{2m}$ where $f(a) = a'$ and $f(\delta) = \delta'$.
Nearly identically to the above case (a),

$$\delta_{x_1j} \equiv a_{x_1j+1} - a_{x_1j} \quad \equiv a'_{x_1j+1} - a'_{x_1j} \equiv a_{x_4j+1} - a_{x_3j} \pmod{2m} \quad (6)$$
$$\delta_{x_2j} \equiv a_{x_2j+1} - a_{x_2j} \quad \equiv a'_{x_2j+1} - a'_{x_2j} \equiv a_{x_3j+1} - a_{x_4j} \pmod{2m} \quad (7)$$

addition of equations (6) and (7) give that

$$a_{x_1j+1} - a_{x_1j} + a_{x_2j+1} - a_{x_2j} \equiv a_{x_3j+1} - a_{x_3j} + a_{x_4j+1} - a_{x_4j} \pmod{2m}$$
$$\Rightarrow \delta_{x_1j} + \delta_{x_2j} \equiv \delta_{x_3j} + \delta_{x_4j} \pmod{2m}$$

Furthermore,

$$\delta_{x_1j-1} \equiv a_{x_1j} - a_{x_1j-1} \quad \equiv a'_{x_1j} - a'_{x_1j-1} \equiv a_{x_4j} - a_{x_3j-1} \pmod{2m} \quad (8)$$

$$\delta_{x_2 j-1} \equiv a_{x_2 j} - a_{x_2 j-1} \quad \equiv a'_{x_2 j} - a'_{x_2 j-1} \equiv a_{x_3 j} - a_{x_4 j-1} \pmod{2m} \quad (9)$$

addition of equations (8) and (9) give that

$$a_{x_1 j} - a_{x_1 j-1} + a_{x_2 j} - a_{x_2 j-1} \equiv a_{x_3 j} - a_{x_3 j-1} + a_{x_4 j} - a_{x_4 j-1} \pmod{2m}$$
$$\Rightarrow \delta_{x_1 j-1} + \delta_{x_2 j-1} \equiv \delta_{x_3 j-1} + \delta_{x_4 j-1} \pmod{2m}$$

It follows that

$$\delta_{x_1 j+1} + \delta_{x_2 j+1} \equiv \delta_{x_3 j+1} + \delta_{x_4 j+1} \pmod{2m}$$

I have been unable to find or produce examples of either of the eventualities (a) or (b) and suspect that this indicates either that the necessary condition that

$$\begin{aligned} \delta_{x_1 j-1} + \delta_{x_2 j-1} &\equiv \delta_{x_3 j-1} + \delta_{x_4 j-1}, \\ \delta_{x_1 j} + \delta_{x_2 j} &\equiv \delta_{x_3 j} + \delta_{x_4 j} \text{ and} \\ \delta_{x_1 j+1} + \delta_{x_2 j+1} &\equiv \delta_{x_3 j+1} + \delta_{x_4 j+1} \end{aligned}$$

are too strenuous to be practically useful. Furthermore, I believe that there is an additional property of these objects with property P_1 that preclude this eventuality.

2. In the case that $a'_{x_2 j} \neq a_{x_1 j}$, as x_3 is arbitrary and as yet unspecified, without loss of generality we label the positions such that $a'_{x_2 j} = a_{x_3 j}$.

$$\begin{aligned} \text{Since } a'_{x_3 j} &= a_{x_1 j} \\ \Rightarrow a'_{x_4 j} &= a_{x_4 j}, \text{ a contradiction, we therefore have that} \\ a'_{x_3 j} &= a_{x_4 j} \text{ and } a'_{x_4 j} = a_{x_1 j}. \end{aligned}$$

We have that

$$\{\delta'_{x_1 j-1}, \delta'_{x_2 j-1}, \delta'_{x_3 j-1}, \delta'_{x_4 j-1}\} = \{\delta_{x_1 j-1}, \delta_{x_2 j-1}, \delta_{x_3 j-1}\delta_{x_4 j-1}\},$$
$$\{\delta'_{x_1 j}, \delta'_{x_2 j}, \delta'_{x_3 j}, \delta'_{x_4 j}\} = \{\delta_{x_1 j}, \delta_{x_2 j}, \delta_{x_3 j}\delta_{x_4 j}\}.$$

All arithmetic in the subscript of x is taken modulo 4.

This give the conditions that for $a \in \{1, \ldots, 4\}$ since

- $\delta'_{x_a j} = \delta_{x_a j} \Rightarrow a'_{x_a j} = a_{x_a j}$ is a contradiction and

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

- $\delta'_{x_a j} = \delta_{x_{a+1} j} \Rightarrow a'_{x_a j+1} = a_{x_{a+1} j+1}$ is a contradiction, then
- $\delta'_{x_a j} = \delta_{x_{a+2} j}$ or $\delta'_{x_a j} = \delta_{x_{a+3} j}$.

Furthermore, if there are some $\delta'_{x_a j} = \delta_{x_{a+2} j}$ and some $\delta'_{x_a j} = \delta_{x_{a+3} j}$ we again reach the contradiction that for some $b \in \{1, \ldots 4\}$, $\delta'_{x_b j} = \delta_{x_b j}$. Hence we have two possibilities for mappings of $\delta'_{x_a j}$. That is, either $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j} = \delta_{x_{a+2} j}$ or $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j} = \delta_{x_{a+3} j}$.

Similarly, we have two possibilities for mappings of $\delta'_{x_a j-1}$. Firstly that $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j-1} = \delta_{x_{a+2} j-1}$ or secondly that $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j-1} = \delta_{x_{a+3} j-1}$.

Combining these two possibilities for each set of variables, we have four cases. (a) is the case that $\delta'_{x_a j} = \delta_{x_{a+2} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+2} j-1}$, (b) is that $\delta'_{x_a j} = \delta_{x_{a+3} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+3} j-1}$, (c) is the case that $\delta'_{x_a j} = \delta_{x_{a+2} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+3} j-1}$ and (d) is the case that $\delta'_{x_a j} = \delta_{x_{a+3} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+2} j-1}$

(a) Suppose that $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j} = \delta_{x_{a+2} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+2} j-1}$.
Then

$$\begin{aligned} -\delta_{x_a j+1} &\equiv \delta'_{x_a j} + \delta'_{x_a j-1} \pmod{2m} \\ &\equiv \delta_{x_{a+2} j} + \delta'_{x_{a+2} j-1} \pmod{2m} \\ &\equiv -\delta_{x_{a+2} j+1} \pmod{2m} \end{aligned}$$

and

$$\begin{aligned} -\delta_{x_{a+1} j+1} &\equiv \delta'_{x_{a+1} j} + \delta' x_{a+1} j - 1 \pmod{2m} \\ &\equiv \delta_{x_{a+3)} j} + \delta' x_{a+3)} j - 1 \pmod{2m} \\ &\equiv -\delta_{x_{a+3} j+1} \pmod{2m} \end{aligned}$$

which is a contradiction as it implies that the multiset $\{1, \ldots, m, m, \ldots, 2m-1\}$ contains four copies of m.

(b) Suppose that $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_a j} = \delta_{x_{a+3} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+3} j-1}$.
Then

$$\begin{aligned} -\delta_{x_a j+1} &\equiv \delta'_{x_a j} + \delta'_{x_a j-1} \pmod{2m} \\ &\equiv \delta_{x_{a+3} j} + \delta'_{x_{a+3} j-1} \pmod{2m} \end{aligned}$$

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c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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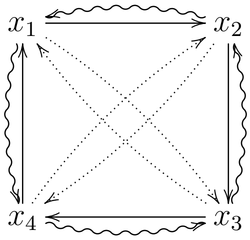
$$\equiv -\delta_{x_{a+3}j+1} \pmod{2m}$$

and

$$\begin{aligned}-\delta_{x_{a+1}j+1} &\equiv \delta'_{x_{a+1}j} + \delta'_{x_{a+1}j-1} \pmod{2m}\\ &\equiv \delta_{x_{a)}j} + \delta'_{x_aj-1} \pmod{2m}\\ &\equiv -\delta_{x_aj+1} \pmod{2m}\end{aligned}$$

which is again a contradiction as it implies that the multiset $\{1, \ldots, m, m, \ldots, 2m-1\}$ contains four copies of m.

(c) Suppose that $\forall a \in \{1, \ldots, 4\}$, $\delta'_{x_aj} = \delta_{x_{a+2}j}$ and $\delta'_{x_aj-1} = \delta_{x_{a+3}j-1}$.
Graphically



where a black vector represents the mapping of $a'_{x_\alpha j} \rightarrow a_{x_{\alpha+1}j}$, a dotty vector the mapping of $\delta'_{x_\alpha j} \rightarrow \delta_{x_{\alpha+2}j}$ and a squiggly vector the mapping of $\delta'_{x_\alpha j-1} \rightarrow \delta_{x_{\alpha+3}j-1}$.
To extend this case, it is apparent that the system of equations

$$\begin{aligned}
\delta_{x_1j} &\equiv a_{x_1j+1} - a_{x_1j} &&\equiv a_{x_3j+1} - a_{x_4j} \pmod{2m} \quad (10)\\
\delta_{x_2j} &\equiv a_{x_2j+1} - a_{x_2j} &&\equiv a_{x_4j+1} - a_{x_1j} \pmod{2m} \quad (11)\\
\delta_{x_3j} &\equiv a_{x_3j+1} - a_{x_3j} &&\equiv a_{x_1j+1} - a_{x_2j} \pmod{2m} \quad (12)\\
\delta_{x_4j} &\equiv a_{x_4j+1} - a_{x_4j} &&\equiv a_{x_2j+1} - a_{x_3j} \pmod{2m} \quad (13)
\end{aligned}$$

give that the addition of (11) to (13) yields the necessary condition that

$$a_{x_2j} + a_{x_4j} \equiv a_{x_1j} + a_{x_3j} \pmod{2m}. \quad (14)$$

Similarly, addition of equations (15) and (17)

$$\begin{aligned}
\delta_{x_1j-1} &\equiv a_{x_1j} - a_{x_1j-1} &&\equiv a_{x_3j} - a_{x_2j-1} \pmod{2m} \quad (15)\\
\delta_{x_2j-1} &\equiv a_{x_2j} - a_{x_2j-1} &&\equiv a_{x_4j} - a_{x_3j-1} \pmod{2m} \quad (16)
\end{aligned}$$

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c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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$$\begin{aligned}
\delta_{x_3 j-1} &\equiv a_{x_3 j} - a_{x_3 j-1} &&\equiv a_{x_1 j} - a_{x_4 j-1} \pmod{2m} &\quad (17)\\
\delta_{x_4 j-1} &\equiv a_{x_4 j} - a_{x_4 j-1} &&\equiv a_{x_2 j} - a_{x_1 j-1} \pmod{2m} &\quad (18)
\end{aligned}$$

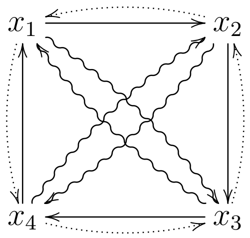
give that

$$a_{x_1 j-1} + a_{x_3 j-1} \equiv a_{x_2 j-1} + a_{x_4 j-1} \pmod{2m}. \tag{19}$$

Subtracting equation (19) from (14) then gives that

$$\delta_{x_1 j-1} + \delta_{x_3 j-1} \equiv \delta_{x_2 j-1} + \delta_{x_4 j-1} \pmod{2m}. \tag{20}$$

(d) Suppose that $\forall a \in \{1, \ldots, 4\}$ $\delta'_{x_a j} = \delta_{x_{a+3} j}$ and $\delta'_{x_a j-1} = \delta_{x_{a+2} j-1}$.
Again we have the diagram that



where a black vector represents the mapping of $a'_{x_\alpha j} \to a_{x_{\alpha+1} j}$, a dotty vector the mapping of $\delta'_{x_\alpha j} \to \delta_{x_{\alpha+3} j}$ and a squiggly vector the mapping of $\delta'_{x_\alpha j-1} \to \delta_{x_{\alpha+2} j-1}$.
To extend this case, it is apparent that the system of equations

$$\begin{aligned}
\delta_{x_1 j} &\equiv a_{x_1 j+1} - a_{x_1 j} &&\equiv a_{x_2 j+1} - a_{x_3 j} \pmod{2m} &\quad (21)\\
\delta_{x_2 j} &\equiv a_{x_2 j+1} - a_{x_2 j} &&\equiv a_{x_3 j+1} - a_{x_4 j} \pmod{2m} &\quad (22)\\
\delta_{x_3 j} &\equiv a_{x_3 j+1} - a_{x_3 j} &&\equiv a_{x_4 j+1} - a_{x_1 j} \pmod{2m} &\quad (23)\\
\delta_{x_4 j} &\equiv a_{x_4 j+1} - a_{x_4 j} &&\equiv a_{x_1 j+1} - a_{x_2 j} \pmod{2m} &\quad (24)
\end{aligned}$$

give that (21) plus (23) yields

$$a_{x_1 j+1} + a_{x_3 j+1} \equiv a_{x_2 j+1} + a_{x_4 j+1} \pmod{2m} \tag{25}$$

as in the previous eventuality.
Similarly, the addition of (26) and (28) of the following equations

$$\begin{aligned}
\delta_{x_1 j-1} &\equiv a_{x_1 j} - a_{x_1 j-1} &&\equiv a_{x_4 j} - a_{x_3 j-1} \pmod{2m} &\quad (26)\\
\delta_{x_2 j-1} &\equiv a_{x_2 j} - a_{x_2 j-1} &&\equiv a_{x_1 j} - a_{x_4 j-1} \pmod{2m} &\quad (27)\\
\delta_{x_3 j-1} &\equiv a_{x_3 j} - a_{x_3 j-1} &&\equiv a_{x_2 j} - a_{x_1 j-1} \pmod{2m} &\quad (28)
\end{aligned}$$

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c/- The University of Melbourne
Victoria 3010 Australia

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Fax: +61 3 9349 4106
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$$\delta_{x_4 j-1} \equiv a_{x_4 j} - a_{x_4 j-1} \qquad \equiv a_{x_3 j} - a_{x_2 j-1} \pmod{2m} \tag{29}$$

yields the necessary condition that

$$a_{x_2 j} + a_{x_4 j} \equiv a_{x_1 j} + a_{x_3 j} \pmod{2m} \tag{30}$$

in this case too.

Subtracting equation (30) from (25) then gives that

$$\delta_{x_1 j-1} + \delta_{x_3 j-1} \equiv \delta_{x_2 j-1} + \delta_{x_4 j-1} \pmod{2m} \tag{31}$$

It is worth noting that cases (c) and (d) are isomorphic where we consider the bijective mapping of the labelling of $C_{j+1} \to C_{j-1}$ and $C_{j-1} \to C_{j+1}$, given that the set $\{C_1, C_2, C_3\}$ is unordered.

For this reason, we will refer to a subset $\{x_1, x_2, x_3, x_4\}$ of an array

$$\begin{array}{c} C_1 \\ \hline C_2 \\ \hline C_3 \end{array}$$

possessing the property that for some and some distinct $j,\ k$,

$$\begin{aligned} a_{x_1 k} + a_{x_3 k} &\equiv a_{x_2 k} + a_{x_4 k} \pmod{2m} \\ a_{x_2 j} + a_{x_4 j} &\equiv a_{x_1 j} + a_{x_3 j} \pmod{2m} \\ \delta_{x_1 j-1} + \delta_{x_3 j-1} &\equiv \delta_{x_2 j-1} + \delta_{x_4 j-1} \pmod{2m} \end{aligned}$$

as having property P_2.

3.4.2 Property P_2

Interestingly, property P_2 is not a sufficient condition for a swap to result in a set with property P_1. That is, we may take a set of mutually nearly orthogonal latin squares and isolate a required set $\{x_1, x_2, x_3, x_4\}$ with property P_2 yet still be unable to produce a further set of by way of deranging these entries to form a second set of mutually nearly orthogonal latin squares from the first. Hence, property P_2 is necessary for property P_1 in our swapped array, but not sufficient. For instance, consider the following set of vectors $\{C_1, C_2, C_3\}$ with proper P_1.

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Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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C_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$C_2 =$	1	3	5	7	9	11	13	0	2	4	6	8	10	12 .
C_3	8	4	1	13	11	2	10	**9**	**0**	7	5	**12**	**3**	6

The set of vectors $\{C_1, C_2, C_3'\}$ also posses property P_1 where C_3' is distinct from C_3 only in positions containing 0, 3, 9 and 12, specifically

C_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$C_2 =$	1	3	5	7	9	11	13	0	2	4	6	8	10	12 .
C_3'	8	4	1	13	11	2	10	**12**	**9**	7	5	**3**	**0**	6

Note that we can label these positions such that $a'_{x_1j} = a_{x_2j} = 12$, $a'_{x_2j} = a_{x_3j} = 3$, $a'_{x_3j} = a_{x_4j} = 0$ and $a'_{x_4j} = a_{x_1j} = 9$ and thus

$$\begin{aligned} a_{x_1j} + a_{x_3j} \equiv a_{x_2j} + a_{x_4j} & \equiv 12 \pmod{2m} \\ \text{and } a_{x_1j-1} + a_{x_3j-1} \equiv a_{x_2j-1} + a_{x_4j-1} & \equiv 10 \pmod{2m} \end{aligned}$$

However, the positions filled in the j^{th} row containing 0, 3, 10 and 13 of C_3 also fulfil the property in that

$$\begin{aligned} a_{x_1j-1} + a_{x_3j-1} \equiv a_{x_2j-1} + a_{x_4j-1} & \pmod{2m} \text{ since} \\ 2 + 7 \equiv 13 + 10 & \pmod{14}, \\ 0 + 13 \equiv 3 + 10 & \pmod{14} \\ \Rightarrow a_{x_2j} + a_{x_4j} \equiv a_{x_1j} + a_{x_3j} & \pmod{2m} \text{ and} \\ 6 + 12 \equiv 11 + 7 & \pmod{2m} \\ \Rightarrow \delta_{x_1j-1} + \delta_{x_3j-1} \equiv \delta_{x_2j-1} + \delta_{x_4j-1} & \pmod{2m} \end{aligned}$$

but no derangement of these entries gives a set possessing property P_1.

This is illustrated by the following list. Note that the equivalences above give that the entry with 13 in C_3' would become 10 or 3.

C_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$C_2 =$	1	3	5	7	9	11	13	0	2	4	6	8	10	12
C_3'	8	4	1	10	11	2	–	12	9	7	5	–	–	6

  $\Rightarrow \Delta_3' = (6, 11, 1, 7, 7, 3, -, 12, -, 2, 5, 13, -, 7)$ ※ (three copies of 7 in Δ_3' not allowable)

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c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
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C_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$C_2 =$	1	3	5	7	9	11	13	0	2	4	6	8	10	12
C_3'	8	4	1	3	11	2	–	12	9	7	5	–	–	6

$\Rightarrow \Delta_3' = (6, 11, 1, 0, 7, 3, -, 12, -, 2, 5, 13, -, 7)$ ※ (0 in Δ_3' not allowable)

3.5 $x = 5$

Consider the sets $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ where

C_1	0	1	2	3	4	5	6	7	8	9
$C_2 =$	1	0	4	8	7	9	2	5	3	6
C_3	5	**7**	**9**	**6**	3	**0**	4	**8**	2	1

and

C_1'	0	1	2	3	4	5	6	7	8	9
$C_2' =$	1	0	4	8	7	9	2	5	3	6
C_3'	5	**8**	**7**	**9**	3	**6**	4	**0**	2	1

.

The sets $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ both possess P_1 and note that in this example with $2m = 10$, $x = 5$ is possible.

3.6 $x = 6$

Consider the sets $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ where

C_1	0	1	2	3	4	5	6	7	8	9
$C_2 =$	1	0	4	7	9	8	2	5	3	6
C_3	**9**	**5**	7	**6**	1	**3**	8	**2**	4	**0**

and

C_1'	0	1	2	3	4	5	6	7	8	9
$C_2' =$	1	0	4	7	9	8	2	5	3	6
C_3'	**5**	**9**	7	**2**	1	**6**	8	**0**	4	**3**

.

The sets $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ both possess P_1 and note that in this example with $2m = 10$, $x = 6$ is possible.

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Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

3.7 $x = 7$

Consider the sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ where

C_1	0	1	2	3	4	5	6	7	8	9
C_2	1	0	4	7	9	8	2	5	3	6
C_3	5	**6**	**9**	2	**0**	**7**	**4**	**8**	**1**	3

and

C'_1	0	1	2	3	4	5	6	7	8	9
C'_2	1	0	4	7	9	8	2	5	3	6
C'_3	5	**9**	**7**	2	**1**	**6**	**8**	**0**	**4**	3

The sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ both possess P_1 and note that in this example with $2m = 10$, $x = 7$ is possible.

3.8 $x = 8$

Consider the sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ where

C_1	0	1	2	3	4	5	6	7	8	9
C_2	1	0	4	7	9	8	2	5	3	6
C_3	5	**6**	**9**	2	**0**	**7**	**4**	**8**	**1**	**3**

and

C'_1	0	1	2	3	4	5	6	7	8	9
C'_2	1	0	4	7	9	8	2	5	3	6
C'_3	5	**7**	**3**	2	**1**	**9**	**8**	**0**	**6**	**4**

The sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ both possess P_1 and note that in this example with $2m = 10$, $x = 8$ is possible.

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3.9 $x = 2m$

Consider the sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ where

$$\begin{array}{c} C_1 \\ \hline C_2 \\ \hline C_3 \end{array} = \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 4 & 7 & 9 & 8 & 2 & 5 & 3 & 6 \\ \hline \mathbf{5} & \mathbf{6} & \mathbf{9} & \mathbf{2} & \mathbf{0} & \mathbf{7} & \mathbf{4} & \mathbf{8} & \mathbf{1} & \mathbf{3} \end{array}$$

and

$$\begin{array}{c} C'_1 \\ \hline C'_2 \\ \hline C'_3 \end{array} = \begin{array}{c|c|c|c|c|c|c|c|c|c} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 4 & 7 & 9 & 8 & 2 & 5 & 3 & 6 \\ \hline \mathbf{9} & \mathbf{5} & \mathbf{7} & \mathbf{6} & \mathbf{1} & \mathbf{3} & \mathbf{8} & \mathbf{2} & \mathbf{4} & \mathbf{0} \end{array}.$$

The sets $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ both possess P_1 and note that in this example with $2m = 10$, $x = 8$ is possible.

4 Reflection property

Claim: For a set $\{C_1, C_2, C_3\}$ with P_1 if $\forall k \in \{1, \ldots, 3\}, i \in \{0, \ldots, 2m-1\}$,

$$a_{ki} + a_{k2m-1-i} = 2m - 1$$

this is equivalent to a similar property of the difference array; that for all $j \in \{1, \ldots, 3\}, i \in \{0, \ldots, 2m-1\}$,

$$\delta_{ki} + \delta_{k2m-1-i} \equiv 0 \pmod{2m}.$$

Proof.

Forward

Consider a set $\{C_1, C_2, C_3\}$ with P_1 and the additional property that $a_{ki} + a_{k2m-1-i} = 2m - 1$ for all $k \in \{1, 2, 3\}, i \in \{0, \ldots, 2m-1\}$.Then as

$$\begin{aligned} \delta_{ki} + \delta_{k2m-1-i} &\equiv a_{k+1i} - a_{k,i} + a_{k+12m-1-i} - a_{k2m-1-i} \pmod{2m} \\ &\equiv (a_{k+1i} + a_{k+12m-1-i}) - (a_{ki} + a_{k2m-1-i}) \pmod{2m} \\ &\equiv 2m - 1 - (2m-1) \pmod{2m} \\ &\equiv 0 \pmod{2m} \end{aligned}$$

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Reverse

Consider our set with the Methods of Difference property and corresponding stepwise difference vectors having the property that for all $k \in \{1, 2, 3\}, i \in \{0, \ldots, 2m-1\}$,

$$\delta_{ki} + \delta_{k2m-1-i} = 0.$$

Firstly we note that already, $\forall i \in \{0, \ldots, 2m-1\}, a_{1i} + a_{22m-1-i} = 2m-1$. Then, we note that by our definition in the case where $k = 1$, then

$$\begin{aligned} a_{2i} - i + a_{22m-1-i} - (2m-1-i) &\equiv 0 \pmod{2m} \text{ therefore} \\ a_{2i} + a_{22m-1-i} &\equiv 2m-1 \pmod{2m} \end{aligned}$$

Similarly, since $\delta_{3i} + \delta_{3(2m-1-i)} = 0$ we have that

$$\begin{aligned} i - a_{3i} + (2m-1-i) - a_{3n-1-i} &\equiv 0 \pmod{2m} \text{ hence} \\ a_{3i} + a_{32m-1-i} &= 2m-1 \end{aligned}$$

5 Potential Application of Sets of Nearly Orthogonal Latin Squares to Latin Hypercube Sampling

A *latin hypercube*, notated LHC, is an $n \times d$ matrix, wherein each column is a permutation of some n set. It is said to have d factors. Hence, simply by transposing an array representing a set of mutually nearly orthogonal latin squares, we obtain a latin hypercube.

Latin hypercube sampling is a method of experimental design often implemented in computer experiments and numerical integration (Tang 2008). It is performed by selecting a latin hypercube design and using the entries of this design to signify at which point on each parameter a sample is taken. Often used as an alternative to random selection, it stratifies sampling over the desired d parameters (Tang 1993).

However, as with all tools used by persons out of the field of their development, the method is prone to deficiencies in implementation. The vast array of latin hypercubes to choose from allows for the selection of less appropriate latin hypercubes. This potentially leads to a decrease in their statistical relevance as an experimental tool. It is known that latin hypercubes with good space-filling properties are beneficial for experimental design (Tang 2008).

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It is also known that latin hypercubes desirable for experimental design have space-filling properties and low correlation between variables.

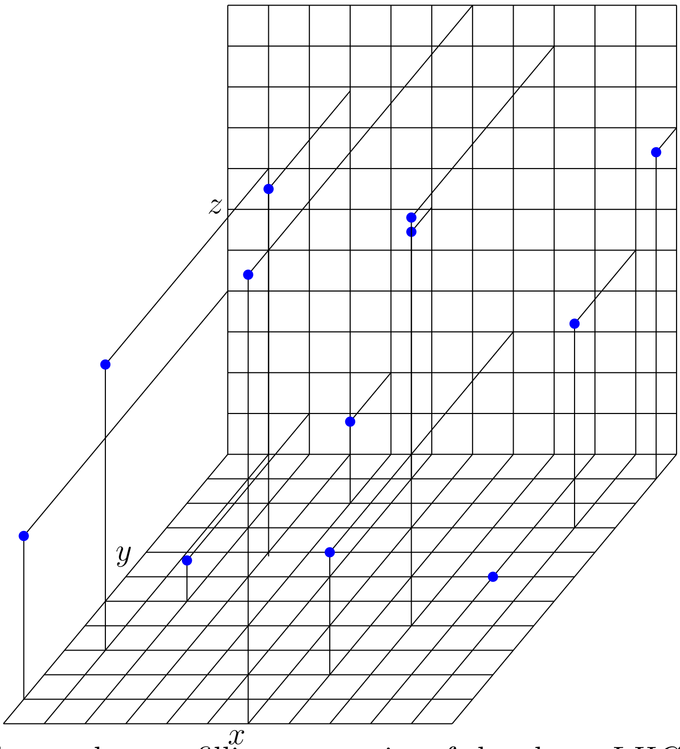
I suggest that sets of mutually nearly orthogonal latin squares could be advantageous for latin hypercube selection.

For instance, we use the following set of mutually nearly orthogonal latin squares to generate a latin hypercube of factor 3.

C_1	0	1	2	3	4	5	6	7	8	9	10	11
C_2 =	1	3	5	7	9	11	0	2	4	6	8	10
C_3	4	7	1	9	2	6	11	3	10	0	5	8

$$LHC = \begin{matrix} 0 & 1 & 4 \\ 1 & 3 & 7 \\ 2 & 5 & 1 \\ 3 & 7 & 9 \\ 4 & 9 & 2 \\ 5 & 11 & 6 \\ 6 & 0 & 11 \\ 7 & 2 & 3 \\ 8 & 4 & 10 \\ 9 & 6 & 0 \\ 10 & 8 & 5 \\ 11 & 10 & 8 \end{matrix}$$

Which gives the following $12 \times 12 \times 12$ design, where the first column represents the x variable, the second column represents y, the third column z and we cut away lines for visibilities sake.



We note the good space-filling properties of the above LHC.

Since latin hypercubes based upon mutually nearly orthogonal latin squares inherit property P_1, we find that the distances between any two points projected on any of the $\binom{d}{2}$ perimeter planes also take the values multiset $\{1, 2, \ldots, m-1, m, m, m+1, \ldots, 2m-2, 2m-1\}$. As shown by Joseph and Hung (2008), such a stratification of distances minimises correlation and yields a LHC with good space-filling properties. The authors do recognise, however, that the inverse relation between correlation and space-filling is not one-to-one. Such designs, therefore, would prove useful for their inherent space-filling properties.

6 Conclusion

Whilst showing great promise for exploitation in computational and physical experimental design, nearly orthogonal latin squares have yet to be fully described. Methods of generating nearly orthogonal latin squares of many orders are yet to be developed, although this is an area of great promise. I point to the work of my supervisor, Dr Diane Donovan, and her co-author Dr Joanne Hall.

Throughout my AMSI vacation research scholarship, I have investigated some properties of mutually nearly orthogonal latin squares. Primarily, I was concerned with the ability to swap certain rows within sets of mutually nearly orthogonal latin squares, producing distinct sets with the same property. I have found that such row exchanges are not possible for swaps of less than four rows. Even in the case that we are swapping four rows, designs that allow this are very rare. The highly restrictive necessary conditions for this case are not always sufficient and no algorithm for identifying swappable rows has been identified. Interestingly, it appears that, beyond four, any number of rows may be exchanged.

The hope was to, by investigating these swaps, enumerate a lower bound on the number of sets of mutually nearly orthogonal latin squares, or a method of their development. I would hope that the work presented here provides some avenues for progress towards these aims. I have no doubt, however, that further investigation of mutually nearly orthogonal latin squares would yield findings fruitful to the development of experimental design and other yet-considered applications.

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