AMSI
AUSTRALIAN MATHEMATICAL SCIENCES INSTITUTE

National Collaboration in the Mathematical Sciences

On Spherical Thrackles

Tim Koussas
Supervisors: Dr. Grant Cairns and Dr. Yuri Nikolayevsky
La Trobe University

February 28, 2012

Let G be an abstract graph. A *thrackle* drawing $\mathcal{T}(G)$ is a drawing of G where every pair of edges in G either meet at an endpoint or properly cross exactly once. A *spherical* thrackle drawing $\mathcal{ST}(G)$ is a thrackle drawing of G on the unit sphere where the edges of G are represented by arcs of great circles.

The class of spherical thrackle drawings is a natural spherical analog of straight-line thrackles drawn on the plane. Despite the similarity, the graphs which can be drawn as spherical thrackles form a larger class than those which can be drawn as straight-line thrackles. Clearly, by central projection, every graph that can be drawn as a straight-line thrackle can also be drawn as a spherical thrackle, but the converse is not true. By the results of Woodall [1], the only cycles which can be drawn as straight-line thrackles are the odd cycles. In comparison, all even cycles other than the 4-cycle can be drawn as spherical thrackles; that is, every cycle that has a general thrackle drawing also has a spherical thrackle drawing. Using an adaptation of Woodall's edge-insertion procedure for spherical thrackles [1], we can obtain from the 6-cycle drawing the rest of the even cycle drawings, as demonstrated in Figure 1.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

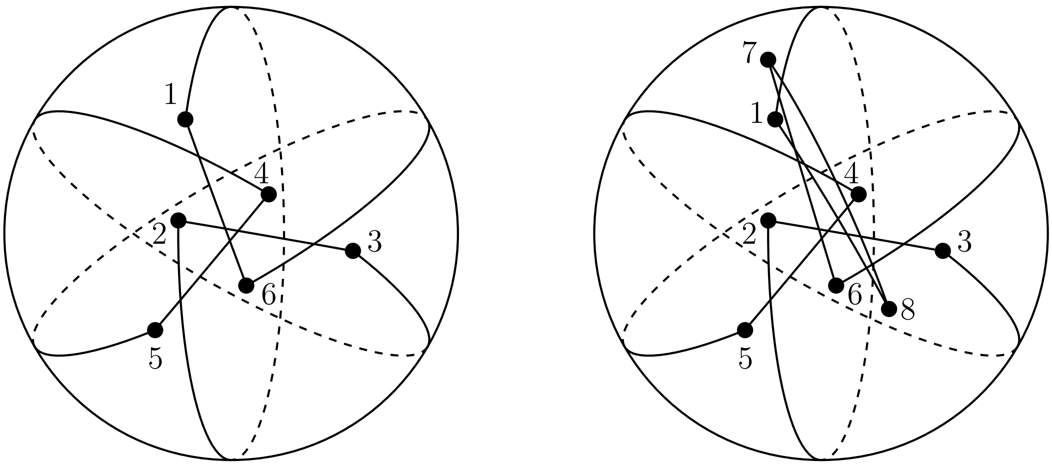


Figure 1: Spherical thrackle drawings of a 6-cycle (left) and an 8-cycle (right).

While our main goal is to prove that the thrackle conjecture holds for spherical thrackles, we begin by proving some results relating to spherical thrackle drawings of cycles, which will assist us in the proof of the main conjecture. First, we require some definitions.

We define the *crossing orientation* of any two directed edges e, f in a similar manner to the vector cross product. To demonstrate this, in Figure 2 we have $\chi(e_3, e_1) = 1$, while $\chi(e_2, e_4) = -1$. A similar definition applies for intersections at endpoints; in Figure 2 we have $\chi(e_1, e_2) = 1$. Note that in general we have $\chi(e, f) = -\chi(f, e)$.

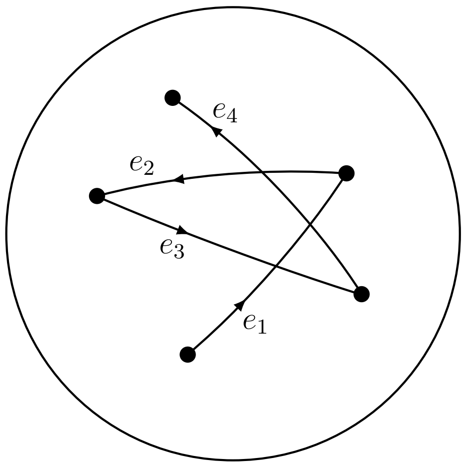


Figure 2: A directed 4-path.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

A (directed) k-path $p = e_1 \dots e_k$ is called *good* if either $\chi(e_{i-1}, e_i) = 1$ for each $i = 2, \dots, k$ or $\chi(e_{i-1}, e_i) = -1$ for each $i = 2, \dots, k$, and is called *bad* otherwise. Similarly, a k-cycle c_k is called good if every directed path in c_k is good, and is called bad otherwise. The path shown in Figure 2 is good.

A *long* edge is an edge whose length is greater than π, while a *short* edge is an edge whose length is less than π.

For any edge e, denote by $\mathbf{C}(e)$ the great circle containing e.

We do not consider cases where two edges lie on the same great circle, so the crossing orientation is well-defined. We also do not consider cases of medium edges, so every edge is either long or short. There is a theorem which allows us to ignore these cases, which we omit.

The Cycle Theorem. *Every spherical thrackle drawing of an n-cycle is good for $n \geq 5$. Moreover, if n is even, its spherical thrackle drawing contains at least one long edge.*

Proof. Let c_n be an n-cycle for some $n \geq 5$. Assume for convenience of notation that $e_k = e_{k+n}$, and choose the direction on c_n in order of increasing edge index.

Suppose c_n is bad. We can assume without loss of generality that there are three adjacent edges e_{j-1}, e_j, e_{j-1} such that $\chi(e_{j-1}, e_j) = 1$ and $\chi(e_j, e_{j+1}) = -1$. Then e_j is short; otherwise, e_{j-1} and e_{j+1} will be forced to lie in the same hemisphere bounded by $\mathbf{C}(e_j)$ in order to intersect, forcing $\chi(e_{j-1}, e_j) = \chi(e_j, e_{j+1})$. We must also have at least one of e_{j-1} and e_{j+1} long, or else they will have no points in common. We assume e_{j-1} and e_{j+1} do not have a common endpoint, since this produces a 3-cycle.

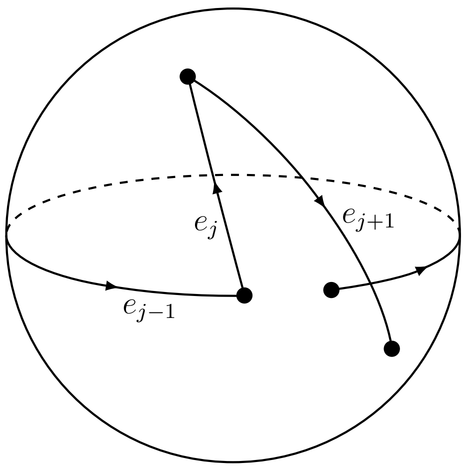


Figure 3: A bad 3-path.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

Up to relabelling and direction change, we have the structure shown in Figure 3, with e_{j+1} possibly long. We see the edge incident to e_{j-1} at its starting point must meet e_j and e_{j+1} at their common endpoint in order to intersect them both, and this produces a 3-cycle, which is a contradiction. Hence, c_n is good.

Now, let $c_{2n} = e_1 \dots e_{2n}$ be a $2n$-cycle for some $n \geq 3$, directed in order of increasing edge index. Suppose that all edges in c_{2n} are short. Denote by $\mathbf{H}$ the hemisphere bounded by $\mathbf{C}(e_1)$ and containing the ending point of e_2. As c_{2n} is good, the starting point of e_{2n} is also in $\mathbf{H}$. As e_3 is short and begins in $\mathbf{H}$, it ends in the other hemisphere $-\mathbf{H}$ (since it must cross e_1 and hence $\mathbf{C}(e_1)$). By similar argument, we see that each even-numbered edge (other than e_{2n}) ends in $\mathbf{H}$, while each odd-numbered edge (other than e_1) ends in $-\mathbf{H}$. But e_{2n-1} must end in $\mathbf{H}$ in order to meet the starting point of e_{2n}, so we have a contradiction. Hence, c_{2n} contains a long edge. □

We now state the following theorem which we will work towards proving.

The Thrackle Conjecture for Spherical Thrackles. *Let G be an abstract graph with n vertices and m edges. If G admits a spherical thrackle drawing $\mathcal{ST}(G)$, then $n \geq m$.*

As is usual in attempting to prove the Thrackle Conjecture, we assume any thrackleable graph G is connected and has no terminal edges, since the existence of any counterexample implies the existence of a counterexample which is connected and has no terminal edges. From the proof of the Cycle Theorem, we observe that, in the context of spherical thrackles, bad paths inevitably produce bad 3-cycles as long as there are no terminal edges. Since we are assuming any counterexample has no terminal edges, this will assist us greatly in proving the conjecture, particularly since it is known that thrackles in general cannot contain more than one 3-cycle [2].

When dealing with thrackles in general as opposed to cycles, we will need to consider vertices with degree other than 2 (i.e. greater than 2 since by assumption there are no terminal edges). Here, a useful notion is that of edge separation. Let v be a vertex and let e be an edge incident to v. For the purpose of illustration, suppose any other edges incident to v are directed away from v. We say e *separates at* v iff there are two edges incident to v other than e which start in opposite hemispheres bounded by $\mathbf{C}(e)$. This is illustrated in Figure 4. If it is understood which vertex is being referred to, or it is irrelevant, we simply say e *separates.*

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

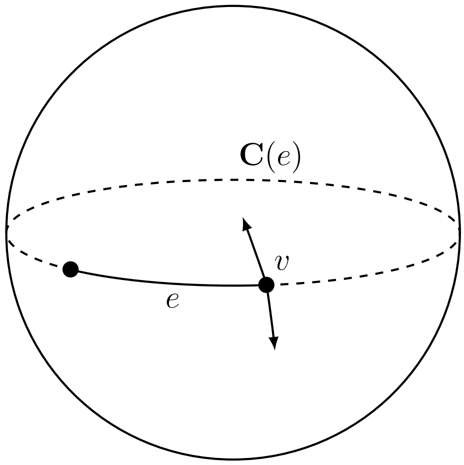


Figure 4: The edge e separates at v.

We have seen that bad paths and medium edges both produce 3-cycles. In particular, the middle edge of a bad 3-path is an edge of a bad 3-cycle. We now explore yet another way of producing bad 3-cycles.

The Separation Lemma. *Let G be a connected graph with no terminal edges. Then any edge in $\mathcal{ST}(G)$ which separates is short and is an edge of a bad 3-cycle.*

Proof. Let e, f, g be edges in $\mathcal{ST}(G)$ which meet at a common vertex v. Assume that g separates at v. Direct e, f towards v and g away from v, and assume without loss of generality that $\chi(e,g) = 1$ and $\chi(f,g) = -1$. Since G has no terminal edges, there is some edge h incident to g at its other endpoint. We have either $\chi(g,h) = 1$ or $\chi(g,h) = -1$. If $\chi(g,h) = 1$, then fgh is a bad 3-path, and if $\chi(g,h) = -1$, then egh is a bad 3-path, so in either case g is the middle edge of a bad 3-path. The middle edge of a bad 3-path is always short, and a bad 3-path produces a bad 3-cycle with the middle edge as one of its sides, so the result follows. $\square$

The Vertex Lemma. *Let G be a connected graph with no terminal edges. If $\deg(v) \geq 3$, then there exists a great circle $\mathbf{C}$ passing through v such that the starting segments of all of the edges incident to v lie in the same hemisphere bounded by $\mathbf{C}$.*

Proof. If no such circle $\mathbf{C}$ exists, then all of the edges must separate; since there are at least three of them, this gives at least two different 3-cycles, which is a contradiction. $\square$

Corollary. *Let G be a connected graph with no terminal edges. Then* $\deg(v) \leq 4$ *for any vertex v in G. Moreover, if* $\deg(v) = 3, 4$, *then v is a vertex of a bad 3-cycle.*

This corollary is an easy consequence of the Vertex Lemma. If we have at least 5 edges starting in the same hemisphere, then there are at least three separating edges; this gives at least two different 3-cycles by the Separation Lemma, which is a contradiction. Hence, $\deg(v) \leq 4$ for any vertex v. If $\deg(v) = 3, 4$, we have at least one separating edge, which must be a side of a bad 3-cycle by the Separation Lemma, so v is a vertex of a bad 3-cycle.

We omit the proof of the next lemma.

The Good Path Lemma. *Let $e_0e_1e_2 \ldots e_{m-1}e_me_{m+1}$ be a simple good path with e_1, e_m long and all other edges short. Then m is odd.*

Now, we restate and prove the following theorem.

The Thrackle Conjecture for Spherical Thrackles. *Let G be an abstract graph with n vertices and m edges. If G admits a spherical thrackle drawing $\mathcal{ST}(G)$, then $n \geq m$.*

Proof. As usual, we assume G is connected and has no terminal edges. Suppose, by way of contradiction, that G has more edges than vertices and that $\mathcal{ST}(G)$ exists.

We prove this theorem in two parts. We first prove that the existence of some $\mathcal{ST}(G)$ with more edges than vertices implies the existence of some $\mathcal{ST}(H)$ which is the drawing of an 8-graph; i.e. H consists of two cycles which share a vertex. In particular, H consists of an even cycle and a bad 3-cycle, with the vertex of degree 4 opposite the long edge of the bad 3-cycle. We then prove that no such graph can be drawn as a spherical thrackle.

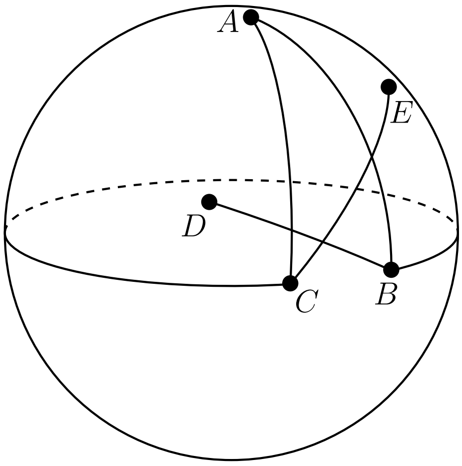
If G has more edges than vertices, then there must be some vertex in G with degree greater than 2. By the corollary of the Vertex Lemma, any such vertex is the vertex of a bad 3-cycle c_3 which must therefore be contained in G. Since G can contain at most one bad 3-cycle, only the vertices of c_3 can have degree greater than 2.

Let $c_3 = ABC$, with A opposite to the unique long edge BC. Let $\mathbf{H}$ be the hemisphere bounded by $\mathbf{C}(BC)$ containing A. We consider the three possible cases (up to symmetry): (1) $\deg(B) > 2$ and $\deg(C) > 2$; (2) $\deg(B) > 2$ and $\deg(C) = 2$; and (3) $\deg(B) = 2$ and $\deg(C) = 2$.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia
Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

Case (1): $\deg(B) > 2$ and $\deg(C) > 2$. If $\deg(B) > 2$, then there is some edge BD incident to B with $D \neq A, C$. Since BC is long, BD is short, so $D \in \mathbf{H}$ since BD must cross AC. BD does not separate, or it produces another bad 3-cycle. If there is some other edge BD' incident to B with $D' \neq A, C, D$, then one of BD, BD' would separate, giving another bad 3-cycle. Hence $\deg(B) = 3$. By symmetry, $\deg(C) = 3$, with some short edge CE incident to C with $E \in \mathbf{H}$. We must then have $\deg(A) = 2$, since if there were some edge AF incident to A with $F \neq B, C$ then AF cannot separate AB, AC, or it produces another bad 3-cycle, and this implies that AF has no points in common with one of BD, CE.

We come to the drawing on the left in Figure 5, from which we can obtain the structure shown on the right in Figure 5 by edge insertion. Since all other vertices have degree 2, we have a cycle sharing the vertex A with a 3-cycle. Since the intersection of the two cycles is a touching intersection, the other cycle must be even by [3].



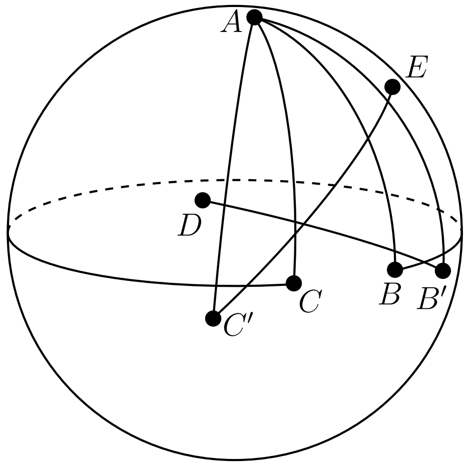


Figure 5: The thrackle drawing on the right is obtained from the drawing on the left.

Case (2): $\deg(B) > 2$ and $\deg(C) = 2$. As in case (1), $\deg(B) = 3$, with some short edge BD incident to B crossing AC. Since $\deg(C) = 2$, and all vertices other than the vertices of c_3 have degree 2, we must have $\deg(A)$ odd so that the degree sum of G is even (which is true of any graph). Hence, $\deg(A) = 3$, since this is the only possible odd degree. The remaining edge AF (with $F \neq B, C$) incident to A must intersect BD and BC, and cannot separate AB, AC, so we get the drawing on the left in Figure 6. By edge insertion, we again obtain an 8-graph consisting of an even cycle and a 3-cycle.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

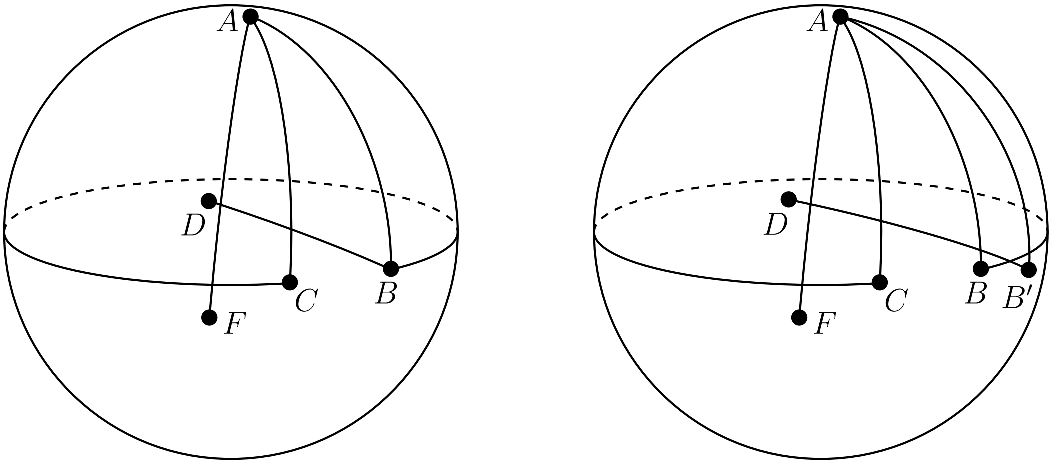


Figure 6: The thrackle drawing on the right is obtained from the drawing on the left.

Case (3): $\deg(B) = 2$ and $\deg(C) = 2$. Since A is the only vertex with degree greater than 2, and the degree sum must be even, we have $\deg(A) = 4$. The other two edges AF, AF' must both cross BC, and since neither of them can separate at A, we get (without loss of generality) the structure shown in Figure 7. We again have an even cycle sharing a vertex with a 3-cycle. This completes the first part of the proof; in each case we have obtained an 8-graph H consisting of an even cycle and a 3-cycle, with the vertex of degree 4 opposite the long edge of the bad 3-cycle.

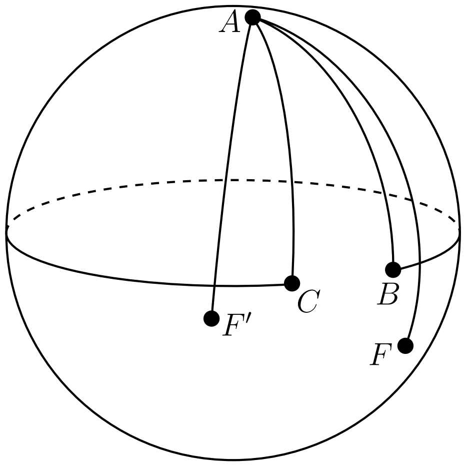


Figure 7: Case (3) gives another 8-graph.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

Now we show that such a graph cannot be drawn in this way. Retain the labelling of the bad 3-cycle with vertices A, B, C, with A opposite the long edge BC. Let $c_{2n} = e_1 \dots e_{2n}$ be the even cycle. Place the direction on c_{2n} in order of increasing edge index, and without loss of generality, assume that $\chi(e_{i-1}, e_i) = 1$ for each $i = 2, \dots, 2n$, and $\chi(e_{2n}, e_1) = 1$ (we know c_{2n} is good from the Cycle Theorem). By swapping the labels of B and C if necessary, assume also that $\chi(AB, BC) = \chi(BC, CA) = 1$. We then get the drawing in Figure 8.

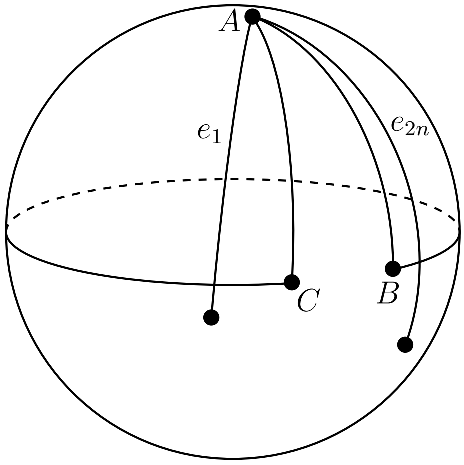


Figure 8: An 8-graph consisting of an even cycle and a 3-cycle.

From Figure 8, we have $\chi(e_{2n}, AB) = \chi(CA, e_1) = 1$.

Now, by the Cycle Theorem, c_{2n} contains at least one long edge. Let e_k and e_ℓ be the first and last edges in c_{2n}, respectively, so $\ell \geq k$.

The path $ABCAe_1 \dots e_{k-1}e_ke_{k+1}$ is good and can be made simple by disconnecting the edges AB and e_{2n} from A. If $e_k = e_{2n}$, we can insert a new edge at the end of e_{2n} close to AB. Hence, by the Good Path Lemma, k is odd. Similarly, the path $e_{\ell-1}e_\ell e_{\ell+1} \dots e_{2n}ABCA$ can be modified to satisfy the assumption of the Good Path Lemma, so ℓ is even. Since e_k and e_ℓ are both long, they are not adjacent, and since k and ℓ have opposite parity, we have $\ell - k \geq 3$.

Finally, consider the path $e_{\ell-1}e_\ell e_{\ell+1} \dots e_{2n}e_1 \dots e_{k-1}e_ke_{k+1}$. Since $\ell - k \geq 3$, the path could possibly be a cycle, but can be made simple by disconnecting e_{k+1} and $e_{\ell-1}$. By the Good Path Lemma $\ell - k$ is even, which is a contradiction, as k and ℓ have opposite parity. This completes the proof. □

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au

References

[1] D. R. Woodall (1971) *Thrackles and deadlock,* Combinatorial Mathematics and its Applications, Proc. 1969 Oxford Combinatorial Conference, Academic Press, 335-347.

[2] G. Cairns and Y. Nikolayevsky (2000) *Bounds for generalized thrackles,* Discrete Comput. Geom., **23**, 191-206.

[3] L. Lovász, J. Pach, M. Szegedy (1997) *On Conway's thrackle conjecture,* Discrete Comput. Geom., **18**, 369-376.

Postal Address: 111 Barry Street
c/- The University of Melbourne
Victoria 3010 Australia

Email: enquiries@amsi.org.au
Phone: +61 3 8344 1777
Fax: +61 3 9349 4106
Web: www.amsi.org.au