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Topology and geometry of Raychandhuri's equation

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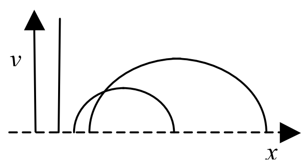
For my summer scholarship, I elected to study with A/Prof. Geoff Prince in the La Trobe Mathematics department in the Differential Geometry area after doing a course on Cosmology with him in third year. The first few weeks were spent studying chapters from Barrett O'Neill's *Elementary Differential Geometry* (1977, Academic Press). I covered the chapters on Covariant Derivatives, Calculus on Surfaces and Manifolds, Shape Operators and Geodesics. I also studied Einstein's summation notation. Some work on Planar Flows with Geoff Prince introduced me to Raychaudhuri's equation.

The equation is: $\mathcal{L}_z(tr(A_z)) = -tr(A_z^2) - tr(\overline{\Phi})$, where *Z* is the tangent vector field to the geodesic congruence, A_Z is the shape map, $\overline{\Phi}$ is the Jacobian endomorphism and $\mathcal{L}_z$ is the Lie-derivative with respect to *Z*. Without going into more detail, this equation is particularly useful in calculating the collapse of congruences of geodesics on manifolds. Hence, for the final few weeks, we calculated the collapse of congruences of geodesics on the *Hyperbolic Plane*, $\mathbf{H}^2$ (also known as *Poincaré's Half-Plane*).

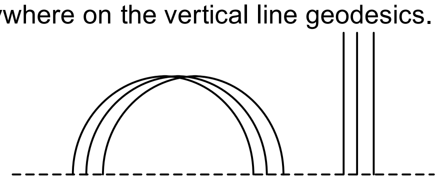
$\mathbf{H}^2$ has the metric $g = ds^2 = (dx^2 + dy^2)/y^2$, which gives the geodesic equation: $d/ds(\dot{x}/y^2) = 0$. This integrates to $\dot{x} = cy^2$. If *c* = 0, then *x* = *const.* which is a vertical line. Otherwise we find $(x-a)^2 + y^2 = 1/c^2$ which is the equation for a semicircle centred at *(x, y) = (a, 0)*.



Geodesics of the Hyperbolic Plane

Using the mathematical software package, *DimSym*, we found the symmetries of the geodesics. We calculated the collapse of congruences under the following symmetries: translation along *x*, $X_1 = \partial/\partial x$; and dilations, $X_2 = x(\partial/\partial x) + y(\partial/\partial y)$. The calculations were messy but the results are easy to interpret.

For the translations along *x*, we found the geodesics collapsed at $\dot{y} = 0$ which is at the top of the semi-circular geodesics and nowhere on the vertical line geodesics. For the dilations about *a*, we found that the geodesics collapsed at $\dot{y}/\dot{x} = y/x$ which is nowhere on the semi-circular geodesics and everywhere on the vertical line geodesics.



Collapse of geodesics in the translation symmetry

Further areas of study would be calculating the collapses for the remaining symmetries and also for the isometric image of $\mathbf{H}^2$, the Hyperbolic Disc, **D**.

The experience was invaluable – apart from learning about an area of maths in which I had a particular interest and exposing me to some of the higher levels of mathematics one works on in postgraduate studies, it also kept my tools sharp in preparation for my honours year. I would recommend the scholarship to any keen maths student.

Richard received an ICE-EM Vacation Scholarship in December 2005.
See www.ice-em.org.au/students.html#scholarships05