THE DEVELOPMENT AND APPLICATION OF A NATURAL DUALITY FOR TERNARY ALGEBRAS

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Abstract. This paper examines ternary algebras, their practical applications and how to translate between the natural dual and the restricted Priestley dual of these algebras. We begin by determining the term functions of the standard ternary algebra as an application of the Baker-Pixley Theorem. This is followed by an overview of some of the practical applications of ternary algebras. We establish the ability to transform both the natural dual of an arbitrary ternary algebra into its restricted Priestley dual, and the restricted Priestley dual of an arbitrary ternary algebra into its natural dual. The translation process is formalized and two applications are demonstrated. This discovery is prefaced by using existing natural dualities theory to establish an optimal natural duality and the restricted Priestley duality of ternary algebras.

1. Overview

This report aims to encompass the objectives described in the initial project description. These outcomes are:

- produce an overview of the many applications of ternary algebras,
- apply the theory of natural dualities to study ternary algebras,
- investigate the extent to which restricted Priestley duality and natural duality can be used in tandem.

Almost all the results in this report are presented without proof. The author and her supervisor will eventually write up all the proofs for publication.

2. Introduction

In classical propositional logic every proposition is either true or false. That is, classical logic has exactly two truth-values. The algebraic counterpart of classical logic is Boolean algebra. On the other hand, any form of logic that allows for more than two truth-values belongs to the realm of non-classical logic. Examples of algebras arising from non-classical logic are Kleene algebras and ternary algebras.

The standard Boolean algebra is given by

$$\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, \neg, 0, 1\rangle.$$

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

	$\neg$
0	1
1	0

Figure 1. The operations of $\underline{\mathbf{2}}$

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The standard Kleene algebra is given by

$$\underline{\mathbf{K}} = \langle \{0, d, 1\}; \vee, \wedge, \neg, 0, 1\rangle,$$

and the standard ternary algebra is given by

$$\underline{\mathbf{3}} = \langle \{0, d, 1\}; \vee, \wedge, \neg, 0, d, 1\rangle.$$

$\vee$	0	d	1
0	0	d	1
d	d	d	1
1	1	1	1

$\wedge$	0	d	1
0	0	0	0
d	0	d	d
1	0	d	1

	$\neg$
0	1
d	d
1	0

FIGURE 2. The operations of $\underline{\mathbf{K}}$ and $\underline{\mathbf{3}}$

A comparison of $\underline{\mathbf{2}}$ with $\underline{\mathbf{K}}$ reveals that $\underline{\mathbf{2}}$ satisfies the *Law of the Excluded Middle*

$$p \vee \neg p \equiv 1$$

while $\underline{\mathbf{K}}$ fails this law as $d \vee \neg d = d$. The only difference between the standard Kleene algebra $\underline{\mathbf{K}}$ and the standard ternary algebra $\underline{\mathbf{3}}$ is that d is included in the type of $\underline{\mathbf{3}}$ as a nullary operation.

We conclude this section with some definitions.

Definition 2.1. A *term* is a "meaningful" expression built from

- operation symbols: $\vee, \wedge, \neg, 0, d, 1$,
- variable symbols: $x_1, x_2, x_3, \ldots$,
- delimiters: $(,), \ldots$.

For example, $((x_1 \vee x_2) \wedge (\neg(x_1 \vee x_2))) \vee d$ is a binary term while $\neg x_1))\wedge$ is not a "meaningful"expression. A *Boolean term* refers to a term that does not involve the operation symbol d.

Every term yields a corresponding *term function* on $\underline{\mathbf{3}}$ and every Boolean term yields a corresponding term function on both $\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}$. For example, the binary Boolean term $\neg(x \vee y) \wedge \neg x$ yields the term functions on $\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}$ represented as 2-valued and 3-valued truth tables in Figure 3.

p	q	$\neg(x \vee y) \wedge \neg x$
0	0	1
0	1	0
1	0	0
1	1	0

p	q	$\neg(x \vee y) \wedge \neg x$
0	0	1
0	d	d
0	1	0
d	0	d
d	d	d
d	1	0
1	0	0
1	d	0
1	1	0

FIGURE 3. $f\colon \underline{\mathbf{2}}^2 \to \underline{\mathbf{2}}$ and $g\colon \underline{\mathbf{3}}^2 \to \underline{\mathbf{3}}$

The term functions on $\underline{\mathbf{3}}$ corresponding to Boolean terms play an important role in applications and are referred to as *B-term functions*. The term function g in Figure 3 is an example of a B-term function of $\underline{\mathbf{3}}$.

Term functions are used in the analysis of circuits. Terms and term functions are relevant tools in several fields including electrical engineering and computer science.

3. Some applications of ternary algebras

We consider some of the applications of ternary algebras. However, the reader should be aware that the author did not possess the necessary expertise in electronics to understand the intricacies of these applications.

Ternary algebras were initially applied by Goto in 1949 to analyze the indefinite behaviours of relay circuits as well as to synthesize such circuits [12, 13]. Other pioneers to apply ternary algebras include Moisil and Roginskii. See references 17, 18 and 30 in [4].

Since then the potential of ternary algebras has been further recognized. In particular, ternary algebras may be applicable to a circuit containing ambiguity either at the input stage or the output stage. For example, Muller used ternary algebras to study transient phenomena in switching circuits [17]. While Mukaidono demonstrated that ternary algebras can be used to design fail-safe logic circuits by letting d correspond to a failure state [16]. Mukaidono also showed that ternary algebras can correct input failures [15, 16]. He motivates the correction of input failures by highlighting that a fail-safe logic circuit capable of self-correcting as many input failures as possible during normal operation will have the advantages of improved safety and reliability and a decreasing of inactive states. In the case of CMOS circuits, an ambiguous output is represented by the value d [4].

Let us consider a specific example of an application of ternary algebras. It was shown by Yoeli and Rinon that ternary algebras could be utilized to detect static hazards in combinational switching circuits [19]. In particular they use B-term functions of $\underline{\mathbf{3}}$. The use of B-term functions is justified as the overall performance (including transient behaviour) of a binary electronic combinational switching circuit composed of AND-, OR- and NOT-logic circuits will be adequately described by the corresponding B-term function [19].

To detect whether a circuit contains a static hazard we simply need to find the corresponding Boolean term, denote all changing inputs by d and determine the resulting value of the B-term function. A circuit containing a hazard will have the output value d [4].

Mukaidono went one step further and demonstrated that various kinds of static hazards contained in combinational switching circuits can be detected and identified by B-term functions [14]. In particular, he derived a method which could algebraically detect all logic hazards contained in the circuits. Mukaidono also pointed out that there were some dynamic hazards which were detectable by B-term functions.

Traditionally, asynchronous circuits have been viewed as difficult to understand and design [5]. Hence many of the digital circuits in use today are synchronous. However, asynchronous circuits have the potential benefits of increased speed and reduced power consumption. In addition to these advantages, the development of several asynchronous design methodologies has made the design of much larger and more complex circuits possible. This is one of the limitations of synchronous circuits; building large complex circuits as synchronous circuits can be challenging.

An important tool used to detect potential timing errors in asynchronous circuits is based on ternary algebras [3, 4, 6]. This tool is known as ternary simulation and was introduced by Eichelberger [11]. The advantages to ternary simulation are:

- unlike binary analysis algorithms which are exponential in the number of state variables, ternary simulation is linear in the number of state variables [4],
- ternary simulation provides information that even the most accurate circuit simulator cannot [3],
- ternary simulation requires only slightly more computational effort than ordinary logic simulation and hence can be used to check large digital systems operating over long input sequences [3].

Bryant applied ternary simulation to a variety of circuits designed in both nMOS and CMOS [3].

Finally, in addition to the above applications of ternary algebras an example of an application outside the realm of electronics and circuit design is the 3-valued attribute exploration created by Burmeister [7]. A more recent application of ternary algebras was by Negeulescu to a framework for modelling interactive systems known as process spaces [4, 18].

4. The term functions of $\underline{\mathbf{3}}$

We can find the term functions of $\underline{\mathbf{3}}$ by applying the following theorem from Baker and Pixley [1] (see also [8]).

We begin with the concept of a lattice-based finite algebra.

Definition 4.1. A finite algebra $\langle M; F\rangle$ is *lattice-based* if there exist binary operations $\vee, \wedge \in F$ such that $\langle M; \vee, \wedge\rangle$ is a lattice.

Theorem 4.2. *If L is any lattice-based finite algebra, then $f\colon L^n \to L$ is a term function of L if and only if f preserves every binary relation r on L such that r is a subalgebra of L^2.*

We will apply this theorem to find the term functions of $\underline{\mathbf{3}}$.

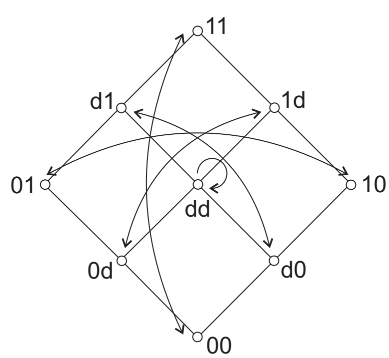

Figure 4. $\underline{\mathbf{3}}^2$

The algebra $\underline{\mathbf{3}}^2$ is shown in Figure 4, and the five subalgebras of $\underline{\mathbf{3}}^2$ are shown in Figure 5. Thus by Theorem 4.2, a map $f\colon \{0, d, 1\}^n \to \{0, d, 1\}$ is a term function

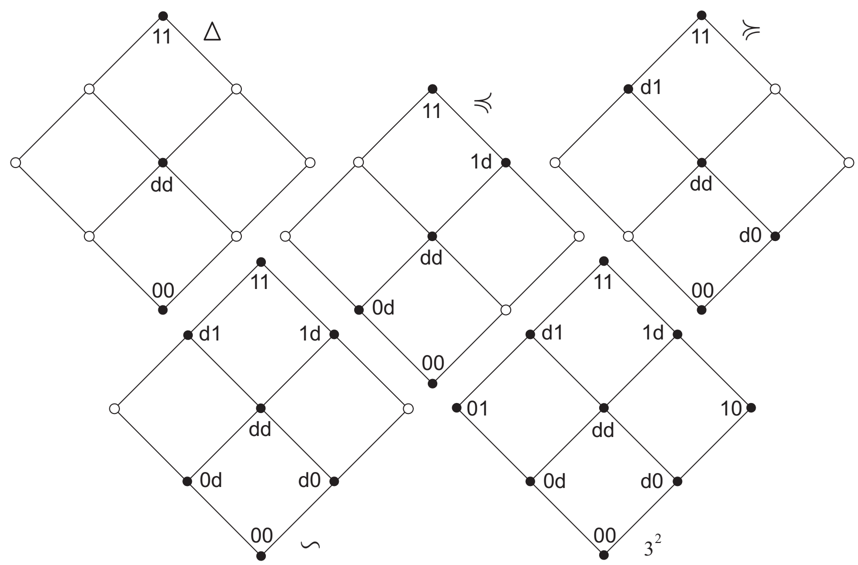

FIGURE 5. The subalgebras of $\underline{\mathbf{3}}^2$

of $\underline{\mathbf{3}}$ if and only if f preserves the binary relations $\Delta, \preccurlyeq, \succcurlyeq, \sim$ and 3^2 shown in Figure 5.

However, this is not an optimal set. It is trivial that every function preserves the binary relations Δ and 3^2 and that if a function preserves a relation r, it also preserves its converse. Thus the binary relations $\Delta, \succcurlyeq$ and 3^2 can be deleted from the list of binary relations that need to be preserved.

We pause at this point to introduce a construct known as the *relational product* of two binary relations r and s:

$$r \cdot s := \{\, (a, b) \in M^2 \mid (\exists c \in M)\ (a, c) \in r \ \ \&\ \ (c, b) \in s \,\}.$$

Notice that

$$\succcurlyeq \cdot \preccurlyeq = \{00, dd, 11, d0, d1\} \cdot \{00, dd, 11, 0d, 1d\} = 3^2 \backslash \{01, 10\} = \sim.$$

That is, $\sim$ is the relational product of $\succcurlyeq$ and $\preccurlyeq$. It is easy to see that if r and s are binary relations on a set M, then every map $f\colon M^n \to M$ that preserves r and s also preserves $r \cdot s$. Thus $\sim$ can also be removed from the list of relations that need to be preserved. Hence the results we obtained by applying Theorem 4.2 can be refined as follows:

Theorem 4.3. *For all $n \geq 1$, a function $f\colon \{0, d, 1\}^n \to \{0, d, 1\}$ is a term function of $\underline{\mathbf{3}}$ if and only if f preserves the binary relation $\preccurlyeq$.*

Observe that $\preccurlyeq$ is an order relation with an order given in Figure 6. This order is often referred to as the *uncertainty order*. It follows that a function

$f\colon \{0,d,1\}^n \to \{0,d,1\}$ is a term function of $\underline{\mathbf{3}}$ if and only if f preserves the uncertainty order. This result was obtained using only algebraic methods by the Japanese electrical engineer, Masao Mukaidono [16].

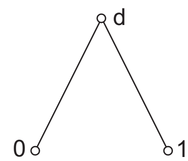

FIGURE 6. The order $\preccurlyeq$ on $\{0,d,1\}$

5. THE NATURAL DUAL AND THE RESTRICTED PRIESTLEY DUAL OF TERNARY ALGEBRAS

5.1. **Natural duality theory—a brief introduction.** We want to find the natural dual for some arbitrary finite ternary algebra. We begin with a brief introduction to the theory of natural dualities. For a more thorough treatment of this topic we refer to Chapters 1 and 2 of [8]. Note that there will be no discussion of topology as we will be working only at the finite level and at this level all topologies that arise are discrete.

The various pieces of the puzzle that we require will be introduced. First, a finite algebra $\underline{\mathbf{M}} := \langle M; F\rangle$ with an underlying set, M, and a set of finitary operations, F. Associated with $\underline{\mathbf{M}}$ is the class $\boldsymbol{\mathcal{A}} := \mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{M}})$ of all algebras that are isomorphic to a subalgebra of a finite direct product of copies of $\underline{\mathbf{M}}$. Just as every finite Boolean algebra is an element of the class $\mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{2}})$, it can be shown that every finite ternary algebra is an element of the class $\boldsymbol{\mathcal{T}} := \mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{3}})$. Second, we begin by choosing R to be a set of finitary relations that are algebraic on $\underline{\mathbf{M}}$. That is, R is a set of finitary relations which form subalgebras of the appropriate power of $\underline{\mathbf{M}}$. We then form the relational structure $\utilde{\mathbf{M}} := \langle M; R\rangle$. (In general, the type of $\utilde{\mathbf{M}}$ may also include operations and partial operations, but they will not be needed in the application to ternary algebras.) Associated with $\utilde{\mathbf{M}}$ is the category of finite structures $\boldsymbol{\mathcal{X}} := \mathbb{ISP}_{\mathrm{f}}(\utilde{\mathbf{M}})$.

Let $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. The natural dual of $\mathbf{A}$ is defined to be

$$D(\mathbf{A}) := \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \subseteq \utilde{\mathbf{M}}^A.$$

That is, the natural dual of $\mathbf{A}$ is the set of all homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{M}}$ with each relation $r \in R$ extended pointwise to $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Thus $D(\mathbf{A}) \in \boldsymbol{\mathcal{X}}$.

Let $\mathbf{X} \in \boldsymbol{\mathcal{X}}$. The natural dual of $\mathbf{X}$ is defined to be

$$E(\mathbf{X}) := \boldsymbol{\mathcal{X}}(\mathbf{X}, \utilde{\mathbf{M}}).$$

In fact, we can say more than this. Because R is a set of *algebraic* relations, it follows that

$$E(\mathbf{X}) := \boldsymbol{\mathcal{X}}(\mathbf{X}, \utilde{\mathbf{M}}) \leqslant \underline{\mathbf{M}}^X.$$

Thus $E(X) \in \boldsymbol{\mathcal{A}}$.

Finally, there is a natural evaluation map $e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$. We say that $\utilde{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$ if $e_{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. As this map is always a one-to-one homomorphism, we only need to show that the map $e_{\mathbf{A}}$ is onto.

5.2. **The natural dual of $\mathcal{T}$.** We apply the development in Section 5.1 to the case where $\underline{\mathbf{M}}$ is the standard ternary algebra $\underline{\mathbf{3}}$. The class $\mathcal{T} := \mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{3}})$ consists of all finite ternary algebras. To find a natural dual for $\mathcal{T}$ we need to find a suitable $\underset{\sim}{\mathbf{3}}$. That is, we need to find a suitable set of finitary algebraic relations on $\underline{\mathbf{3}}$. We will apply a special case of the NU-Duality Theorem from Davey and Werner [10] (see also [8]).

Theorem 5.1. *If $\underline{\mathbf{M}}$ is any lattice-based finite algebra, then $\underset{\sim}{\mathbf{M}} := \langle M; \mathbb{S}(\underline{\mathbf{M}})^2\rangle$ yields a duality on $\mathcal{A}$.*

Thus, by Figure 5, the structure $\underset{\sim}{\mathbf{3}}' = \langle\{0, d, 1\}; \Delta, \preccurlyeq, \succcurlyeq, \sim, 3^2\rangle$ yields a natural duality on $\mathcal{T}$.

5.3. **Optimizing the duality.** A minimal set is easier to work with. So our aim is to find a set of relations that entails the relations $\Delta, \preccurlyeq, \succcurlyeq, \sim$ and 3^2. Admissible constructs include trivial relations and permutation (converse relations fall under this category) but does not include relation product [8, pp. 55–59]. Thus our set of relations includes $\preccurlyeq$ and $\sim$ so far.

It remains to see whether either of these relations, $\preccurlyeq$ or $\sim$, can be removed from the set. We will, first, define what it means for an arbitrary $\underset{\sim}{\mathbf{M}}$ not to yield a duality on $\mathcal{A}$ and then return to our case. Also at this point we introduce some additional notation. We shall denote the algebra whose underlying set is the algebraic relation r by $\mathbf{A}(r)$. Observe that every algebraic relation is the underlying set of an algebra.

Let R_ω denote the set of all finitary algebraic relations on the algebra $\underline{\mathbf{M}}$.

$$
\begin{aligned}
&\underset{\sim}{\mathbf{M}} := \langle M; R\rangle \text{ does not yield a duality on } \mathcal{A} := \mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{M}})\\
\iff& (\exists \mathbf{A} \in \mathcal{A})\ e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) \text{ is not an isomorphism}\\
\iff& (\exists \mathbf{A} \in \mathcal{A})\ e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) \text{ is not onto}\\
\iff& (\exists \mathbf{A} \in \mathcal{A})(\exists \alpha \in ED(\mathbf{A}))(\forall a \in A)\ e_{\mathbf{A}}(a) \neq \alpha\\
\iff& (\exists \mathbf{A} \in \mathcal{A})(\exists \alpha\colon D(\mathbf{A}) \to M)\ \alpha \text{ preserves the relations in R)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \&\ \ (\forall a \in A)\ e_{\mathbf{A}}(a) \neq \alpha\\
\iff& (\exists \mathbf{A} \in \mathcal{A})(\exists \alpha\colon D(\mathbf{A}) \to M)\ \alpha \text{ preserves the relations in R}\\
&\qquad\qquad\qquad\qquad\qquad \&\ \ (\exists s \in R_\omega)\ \alpha \text{ doesn't preserve } s.
\end{aligned}
$$

Note that the only equivalence that is not simply by definition is the last which holds by the Brute Force Theorem [8, 2.3.1].

Now assume that $R \cup \{s\}$ yields a duality on $\mathcal{A}$. Then

$$
\begin{aligned}
&\underset{\sim}{\mathbf{M}} := \langle M; R\rangle \text{ does not yield a duality on } \mathcal{A}\\
\iff& (\exists \mathbf{A} \in \mathcal{A})(\exists \alpha\colon D(\mathbf{A}) \to M)\ \alpha \text{ preserves the relations in } R\\
&\qquad\qquad\qquad\qquad\qquad\qquad \&\ \ \alpha \text{ does not preserve } s\\
\iff& (\exists \alpha\colon D(\underline{\mathbf{s}}) \to M)\ \alpha \text{ preserves the relations in } R\\
&\qquad\qquad\qquad\qquad\qquad\qquad \&\ \ \alpha \text{ does not preserve } s.
\end{aligned}
$$

The second last equivalence holds because $R \cup \{s\}$ yields a duality on $\mathcal{A}$, by assumption, and the final equivalence holds by the Test Algebra Lemma [8, 8.1.3].

Hence if we can find a function $\varphi\colon \mathcal{T}(\mathbf{A}(\sim), \underline{\mathbf{3}}) \to \{0, d, 1\}$ that preserves $\preccurlyeq$ but not $\sim$ and a function $\delta\colon \mathcal{T}(\mathbf{A}(\preccurlyeq), \underline{\mathbf{3}}) \to \{0, d, 1\}$ that preserves $\sim$ but not $\preccurlyeq$ then this is an optimal set.

The set $\mathfrak{T}(\mathbf{A}(\sim), \underline{\mathbf{3}})$ contains the homomorphisms x_1 and x_2 shown in Figure 7. Observe that $x_1 \sim x_2$ and $x_2 \sim x_1$, but $x_1 \not\preccurlyeq x_2$ and $x_2 \not\preccurlyeq x_1$.

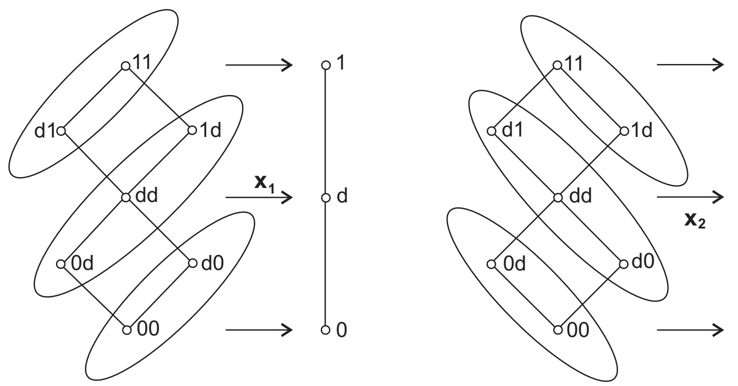

FIGURE 7. The elements of $\mathfrak{T}(\mathbf{A}(\sim), \underline{\mathbf{3}})$

The function $\varphi\colon \mathfrak{T}(\mathbf{A}(\sim), \underline{\mathbf{3}}) \to \{0, d, 1\}$ that maps $x_1 \mapsto 0$ and $x_2 \mapsto 1$ preserves $\preccurlyeq$ but not $\sim$ since $x_1 \sim x_2$ but $\varphi(x_1) = 0 \nsim 1 = \varphi(x_2)$.

The set $\mathfrak{T}(\mathbf{A}(\preccurlyeq), \underline{\mathbf{3}})$ contains the homomorphisms x_3 and x_4 shown in Figure 8. Observe that $x_3 \sim x_4$, $x_4 \sim x_3$ and $x_3 \preccurlyeq x_4$ but $x_4 \not\preccurlyeq x_3$.

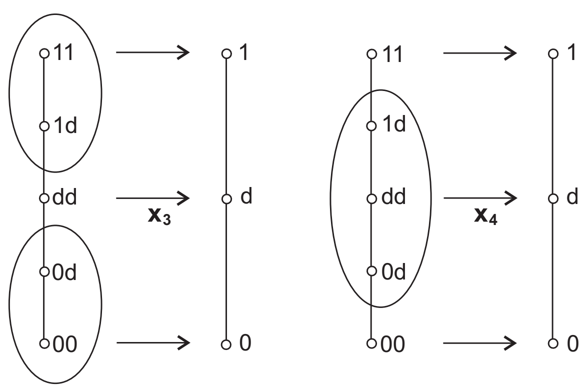

FIGURE 8. The elements of $\mathfrak{T}(\mathbf{A}(\preccurlyeq), \underline{\mathbf{3}})$

The function $\delta\colon \mathfrak{T}(\mathbf{A}(\sim), \underline{\mathbf{3}}) \to \{0, d, 1\}$ that maps $x_3 \mapsto d$ and $x_4 \mapsto 0$ preserves $\sim$ (as $x_3 \sim x_4$ and $\delta(x_3) = d \sim 0 = \delta(x_4)$) but not $\preccurlyeq$ since $x_3 \preccurlyeq x_4$ but $\delta(x_3) = d \not\preceq 0 = \delta'(x_4)$.

Thus we have found the functions we were looking for and so we can conclude that $\underset{\sim}{\mathbf{3}} := \langle \{0, d, 1\}; \preccurlyeq, \sim \rangle$ yields an optimal duality on $\mathfrak{T}$.

Therefore,

$$(\forall \mathbf{A} \in \mathfrak{T})\ D(\mathbf{A}) = \mathfrak{T}(\mathbf{A}, \underline{\mathbf{3}}) \in \mathfrak{X}$$

and

$$\mathbf{A} \cong ED(\mathbf{A}) = \mathfrak{X}(D(\mathbf{A}), \underset{\sim}{\mathbf{3}}).$$

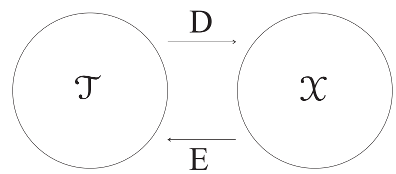

FIGURE 9. The natural dual of $\mathcal{T}$

It would be a useful exercise to find an axiomatization for the natural dual of $\mathcal{T}$ as was done for the natural dual of Kleene algebras by Davey and Werner [10] (see also [8]).

5.4. **Restricted Priestley duality theory.** Now we want to find the restricted Priestley dual for an arbitrary finite ternary algebra. Again, we begin by introducing some basic background theory and the pieces of the puzzle.

Initially we will be moving between the class $\mathcal{D}_{01}$ of finite, bounded distributive lattices and the class $\mathcal{P}$ of finite ordered sets. Let $\underline{\mathbf{D}} := \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ be the two-element bounded distributive lattice and $\underset{\sim}{\mathbf{D}} := \langle \{0,1\}; \leqslant\rangle$ the two-element ordered set with $0 < 1$. Then

$$(\forall \mathbf{L} \in \mathcal{D}_{01})\ H(\mathbf{L}) := \mathcal{D}_{01}(\mathbf{L}, \underline{\mathbf{D}}) \text{ and } (\forall \mathbf{P} \in \mathcal{P})\ K(\mathbf{P}) := \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{D}}).$$

Just as $D(\mathbf{A})$, for all $\mathbf{A} \in \mathcal{T}$, inherits its structure from $\underset{\sim}{\mathbf{3}}$, the Priestley dual $H(\mathbf{L})$ inherits its order from $\underset{\sim}{\mathbf{D}}$. Thus $H(\mathbf{L}) \in \mathcal{P}$. At this point note that, as $D(\mathbf{L}) \cong^{\partial} \mathcal{J}(\mathbf{L})$ and $E(\mathbf{P}) \cong^{\partial} \mathcal{O}(\mathbf{P})$, it is common to work with the join-irreducible elements and simply flip things upside down or in the case of the double dual do nothing as the two $\cong^{\partial}$s cancel each other out.

In order to be able to determine the restricted Priestley dual of a ternary algebra it remains to discuss how to deal with the unary operation $\neg$. This operation can be viewed as the homomorphism $\neg\colon \mathbf{A} \to \mathbf{A}^{\partial}$. The restricted Priestley dual encodes $\neg$ by the map g which will be discussed next.

The restricted Priestley dual of $\neg$ is the order-reversing map

$$g := H(\neg)\colon H(\mathbf{A}^{\partial}) \to H(\mathbf{A}).$$

As $\mathrm{H}(\mathbf{A})^{\partial} \cong H(\mathbf{A}^{\partial})$, the map g can also be expressed as

$$H(\neg)\colon H(\mathbf{A})^{\partial} \to H(\mathbf{A}).$$

An explicit formula for g will be given below.

5.5. **The restricted Priestley dual of $\mathcal{T}$ via the restricted Priestley dual of $\mathcal{K}$.** Recall that

$$\underline{\mathbf{3}} := \langle \{0,d,1\}; \vee, \wedge, \neg, 0, d, 1\rangle \quad \text{and} \quad \underline{\mathbf{K}} := \langle \{0,d,1\}; \vee, \wedge, \neg, 0, 1\rangle.$$

So every ternary algebra is in fact a Kleene algebra. We will build on what is already known about Kleene algebras.

Davey and Priestley have previously established the restricted Priestley dual of $\mathcal{K}$, where $\mathcal{K} := \mathbb{ISP}_{\mathrm{f}}(\underline{\mathbf{K}})$ is the class of all finite Kleene algebras [9] (see also [8]). We begin there.

Let $\mathbf{Y} = \langle Y; g, \leqslant\rangle$ where $g : Y \to Y$ and $\leqslant$ is an order on Y. Then $\mathbf{Y}$ is the restricted Priestley dual of a Kleene algebra, $\mathbf{A} \in \mathcal{K}$, if and only if

(1) g is order-reversing,

(2) $g(g(y)) = y$, for all $y \in Y$, and
(3) for all $y \in Y$, we have $y \leqslant g(y)$ or $y \geqslant g(y)$.

Define $\boldsymbol{\mathcal{Y}}$ to be the class of all structures $\mathbf{Y} = \langle Y; g, \leqslant\rangle$ that satisfy (1)–(3) above. For each $\mathbf{A} \in \boldsymbol{\mathcal{K}}$ define the *restricted Priestley dual* of $\mathbf{A}$ to be

$$H(\mathbf{A}) := \langle \boldsymbol{\mathcal{D}}(\mathbf{A}, \underline{\mathbf{D}}); g, \leqslant\rangle.$$

Here the order is defined pointwise from $\underset{\sim}{\mathbf{D}}$ and the map g is defined by

$$g(y) := c \circ y \circ f,$$

where $c\colon \{0,1\} \to \{0,1\}$ is the Boolean complementation operation and $f(a) := \neg a$, is the negation operator of the algebra $\mathbf{A}$.

To find the restricted Priestley dual of $\boldsymbol{\mathcal{T}}$ we simply need to encode the nullary operation d. We put an additional restriction on g. This restriction is given by

(4) $(\forall y)\ g(y) \neq y$.

We can now define the restricted Priestley dual of $\mathbf{A} \in \boldsymbol{\mathcal{T}}$. Let $\boldsymbol{\mathcal{Z}}$ be the class of all structures $\mathbf{Z} = \langle Y; g, \leqslant\rangle$ that satisfy (1)–(4) above. For each $\mathbf{A} \in \boldsymbol{\mathcal{T}}$, define the *restricted Priestley dual* of $\mathbf{A}$ to be

$$H(\mathbf{A}) := \langle \boldsymbol{\mathcal{D}}(\mathbf{A}, \underline{\mathbf{D}}); g, \leqslant\rangle\ .$$

It should be pointed out that H was initially moving between $\boldsymbol{\mathcal{D}}_{01}$ and $\boldsymbol{\mathcal{P}}$. However, in the restricted Priestley duality for finite ternary algebras, H is now moving between $\boldsymbol{\mathcal{T}}$ and $\boldsymbol{\mathcal{Z}}$.

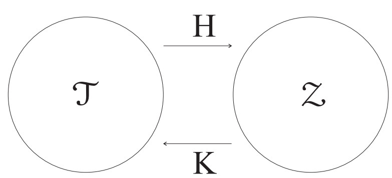

Figure 10. The restricted Priestley dual

6. Translating between the two duals

At this stage we have established both the natural dual and the restricted Priestley dual of $\boldsymbol{\mathcal{T}}$. It remains to work out how to move between these two duals. See Figure 11.

We will begin with translating from the natural dual to the restricted Priestley dual. First, we give an informal description of the translation process. We formalize this description in the next paragraph and then apply the translation to $D(\mathbf{A}(\sim))$ to find $H(D(\mathbf{A}(\sim)))$.

Let $\mathbf{A}$ be a finite ternary algebra and consider the natural dual, $D(\mathbf{A})$, ordered by $\preccurlyeq$. To transform $D(\mathbf{A})$ into the restricted Priestley dual of $\mathbf{A}$, first find its order-theoretic dual, $D(\mathbf{A})^{\partial}$. Then take the disjoint union of $D(\mathbf{A})$ and $D(\mathbf{A})^{\partial}$. Finally, the order-relation between $D(\mathbf{A})$ and $D(\mathbf{A})^{\partial}$ is given by $\sim$. We will now formally define the translation of the natural dual to the restricted Priestley dual.

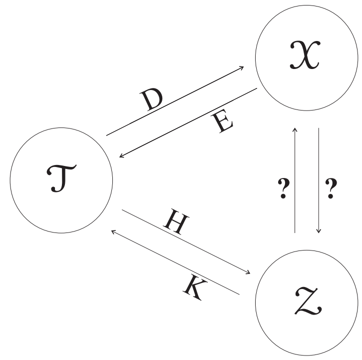

FIGURE 11. The last piece of the puzzle

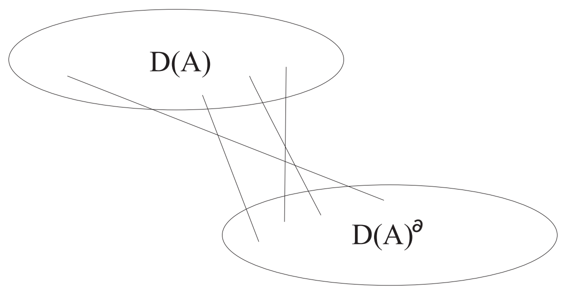

FIGURE 12. $T(D(\mathbf{A}))$

Given $\mathbf{A}$ and its natural dual $D(\mathbf{A}) \in \boldsymbol{\mathfrak{X}}$, define the structure

$$T(D(\mathbf{A})) := \langle D(\mathbf{A}) \times \{1\} \cup D(\mathbf{A}) \times \{0\}; \leqslant, g\rangle$$

where $\leqslant$ is defined by

$$\begin{aligned}(x,i) \leqslant (y,j) \iff & i = j = 1 \ \& \ x \preccurlyeq y \\ & \text{or } i = j = 0 \ \& \ x \succcurlyeq y \text{ or } i = 0 \ \& \ j = 1 \ \& \ (y,x) \in \sim\end{aligned}$$

and g is given by

$$g((x,1)) = (x,0) \quad \& \quad g((x,0)) = (x,1).$$

Then

$$T(D(\mathbf{A})) \cong H(\mathbf{A}) \in \boldsymbol{\mathfrak{Z}}.$$

See Figure 12. Note that

$$(*) \qquad |H(\mathbf{A})| = |T(D(\mathbf{A}))| = 2|D(\mathbf{A})|.$$

Let us look at an example. Consider the ternary algebra $\mathbf{A}(\sim)$. We want to transform $D(\mathbf{A}(\sim))$ into $H(\mathbf{A}(\sim))$. First, the restricted Priestley dual of $\mathbf{A}(\sim))$ is $H(\mathbf{A}(\sim)) = \boldsymbol{\mathfrak{J}}(\mathbf{A}(\sim))$. See Figure 13—the labelling of the elements comes from Figure 5.

Second, the natural dual of $\mathbf{A}(\sim)$ is $\boldsymbol{\mathfrak{T}}(\mathbf{A}(\sim), \underline{\mathbf{3}})$, that is the set of all homomorphisms from $\mathbf{A}(\sim)$ to $\underline{\mathbf{3}}$. See Figure 14. It is easy to see that if we impose $\preccurlyeq$ pointwise on the set $\{x_1, x_2\}$ we get the two-element antichain and if we impose $\sim$

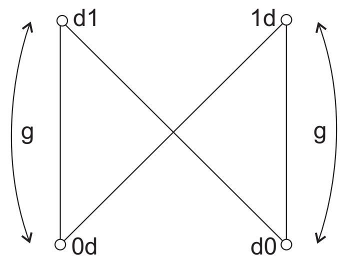

FIGURE 13. The restricted Priestley dual of $\mathbf{A}(\sim)$

point-wise on $\{x_1, x_2\}$ we get $\{x_1, x_2\}^2$, that is the relation relating everything to everything.

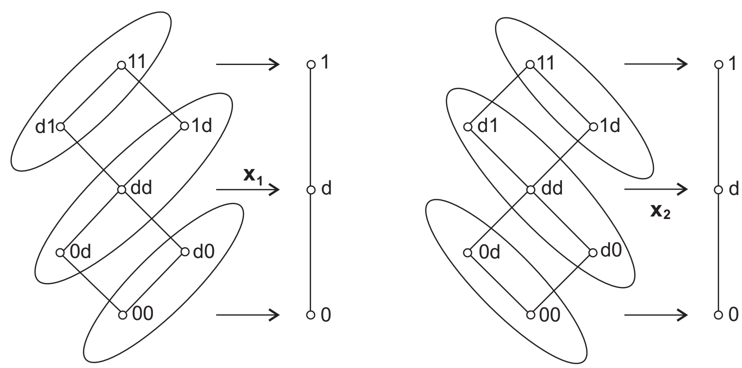

FIGURE 14. The homomorphisms from $\mathbf{A}(\sim)$ to $\underline{\mathbf{3}}$

Recall that we are trying to construct $H(\mathbf{A}(\sim))$. Given $\preccurlyeq^{D(\mathbf{A}(\sim))}$ and $\sim^{D(\mathbf{A}(\sim))}$, we can construct $H(\mathbf{A}(\sim))$ as shown in Figure 15. For another example of the translation process, T, see Figure 17.

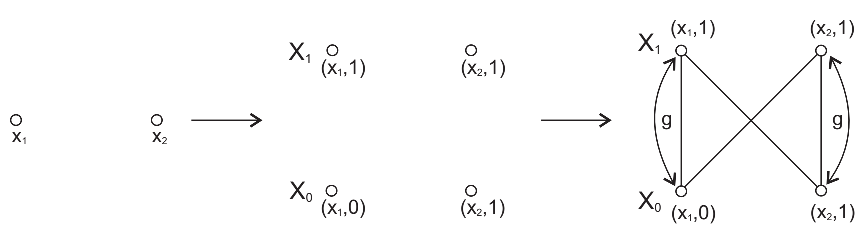

FIGURE 15. $D(\mathbf{A}(\sim)) \rightsquigarrow H(\mathbf{A}(\sim))$

Now we want to start with the restricted Priestley dual and translate to the natural dual. By Axioms (3) and (4) on page 10, we have

$$H(\mathbf{A}) = \mathfrak{D}_{01}(\mathbf{A}, \underline{\mathbf{D}}) = X_1 \cup X_0 \in \mathfrak{Z},$$

where

$$X_1 = \{\, x \mid x > g(x) \,\} \text{ and } X_0 = \{\, x \mid x < g(x) \,\}.$$

Now consider only $\langle X_1; \leqslant\rangle$. Then

$$S(H(\mathbf{A})) := \langle X_1; \leqslant\rangle \cong \langle \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{3}}); \preccurlyeq\rangle \in \boldsymbol{\mathfrak{X}}.$$

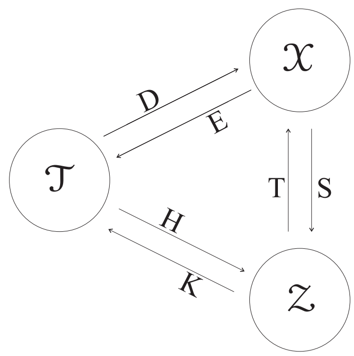

FIGURE 16. The complete picture

6.1. **An application of the translation.** Let $\mathbf{F}_{\mathfrak{T}}(n)$ denote the free ternary algebra on n generators, that is, the subalgebra of $\underline{\mathbf{3}}^{(3^n)}$ whose underlying set is the set of n-ary term functions on $\underline{\mathbf{3}}$. The following lemma is a standard result from the theory of natural dualities (see [8, 2.24]).

Lemma 6.1. $D(\mathbf{F}_{\mathfrak{T}}(n)) \cong \underset{\sim}{\mathbf{3}}^n$.

Thus by Lemma 6.1 and (∗) on page 11, we have

$$|\mathcal{J}(\mathbf{F}_{\mathfrak{T}}(n))| = |H(\mathbf{F}_{\mathfrak{T}}(n))| = 2 \times 3^n.$$

This result was first obtained by Balbes [2] via completely different methods. We can also determine the order on the set of join-irreducible elements of the free-algebra on n-generators:

$$\mathcal{J}(\mathbf{F}_{\mathfrak{T}}(n)) = H(\mathbf{F}_{\mathfrak{T}}(n)) \cong T(D(\mathbf{F}_{\mathfrak{T}}(n))) \cong T(\underset{\sim}{\mathbf{3}}^n).$$

See Figure 17 and Figure 18 for an example. As illustrated in Figures 17 and 18, when working on the full power the minimal elements of $D(\mathbf{A})$ cover the corresponding maximal elements of $D(\mathbf{A})^{\partial}$. The diagram in Figure 18 was first obtained by Balbes using a completely different method which involved a detailed syntactic analysis of ternary algebra terms [2].

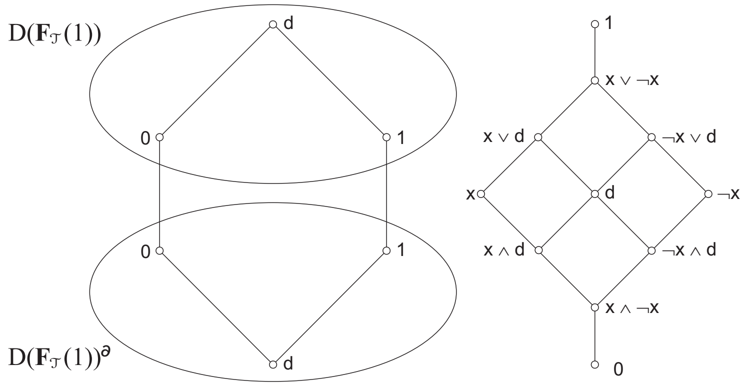

FIGURE 17. $\mathfrak{J}(\mathbf{F}_{\mathcal{T}}(1)) = H(\mathbf{F}_{\mathcal{T}}(1))$ and $\mathbf{F}_{\mathcal{T}}(1)$

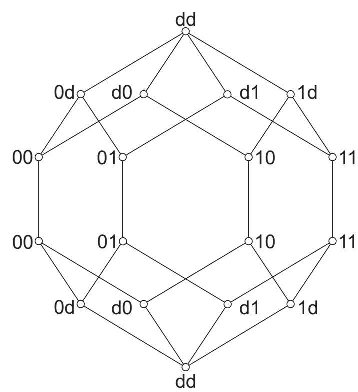

FIGURE 18. $\mathfrak{J}(\mathbf{F}_{\mathcal{T}}(2)) = H(\mathbf{F}_{\mathcal{T}}(2))$

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