Boundedness of Calderón-Zygmund singular integral operators

Anthony Wong
Department of Mathematics
Macquarie University

The aim of my project was to study boundedness of singular integral operators which are usually known as Calderón-Zygmund singular integral operators on L^p spaces of Euclidean spaces. This topic is part of a much wider subject in mathematics called harmonic analysis, which has been a very active area of research and plays an important role in studying solutions of partial differential equations. For this project I investigated two main texts. The first was the book 'Singular Integrals and Differentiability Properties of Functions' by Elias M. Stein. In this book, I focused on chapters 1, 2 and parts of 3. The other text was the paper 'Singular integral operators with non-smooth kernels on irregular domains' by Xuan Duong and Alan MacIntosh. In this paper, I focused on sections 1 and 2.

I discovered that boundedness of operators comes in two forms, strong type and weak type. Strong type implies weak type but the reverse does not hold generally. I learnt about the Classical Marcinkiewicz Interpolation Theorem, which states that if a sublinear operator is of both weak type (p,p) and weak type (q,q) for $1 < p < q \leq \infty$, then it is strong type on (r,r) for all $p < r < q$. An important part of my research was studying the Calderón-Zygmund decomposition, which decomposes a function as the sum of a "good" function and a "bad" function. The Calderón-Zygmund decomposition is regarded as one of the most influential results in mathematics.

I spent time exploring the subject of singular integrals. The study of the existence of singular integrals and their boundedness properties is known as the Calderón-Zygmund theory, which has been the object of intensive study since its origin in the 1950's. A singular integral is an integral operator $T(f)(x) = \int K(x,y)f(y)dy$, whose associated kernel function K is singular along the diagonal $x = y$. A Calderón-Zygmund kernel satisfies 3 conditions: a size, cancellation, and smoothness condition (the last one is also called the Hörmander condition). A singular integral associated to a Calderón-Zygmund kernel is called a Calderón-Zygmund singular integral operator when it is bounded on L^2 (i.e. strong type (2,2)). It can be shown that a Calderón-Zygmund singular integral operator is bounded on L^p (i.e. strong type (p,p)) for all $1 < p \leq \infty$ and is of weak type (1,1). The proof utilizes the Calderón-Zygmund decomposition and the Marcinkiewicz Interpolation

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Theorem. In the paper that I studied it showed that assuming boundedness on L^2, boundedness on L^p for $1 < p \leq \infty$ and weak-type (1,1) can still be achieved when the Hörmander condition is weakened.

As an application, I studied the Hilbert and Riesz transform. Calderón-Zygmund singular integral operator is a generalization of the Hilbert and Riesz transform. The Riesz transform is a generalization of the Hilbert transform to n dimension. They are both bounded on L^p (i.e. strong type (p,p)) for $1 < p \leq \infty$ and weak-type (1,1).

Overall, the project was an interesting and valuable experience. The Big Day In gave me a great opportunity to present my findings and see the wide range of unique and fascinating research, both from mathematics as well as other science disciplines.

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