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Some Configurations of an Affine Space and the game of SET

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1 Introduction

The idea of embedding structures in affine spaces has a rich history in finite geometry, spanning over 60 years. In this project, we take a look at some of these embeddings in the affine space $AG(4,3)$ and their corresponding representations in the game of SET[1].

1.1 The History and Rules of SET

1.1.1 History

The card game SET has its origins in an entirely unexpected context. While studying epilepsy in German Shepherds at Cambridge University in 1974, geneticist Marsha J. Falco decided to keep track of genes by representing them using different shapes drawn on index cards. She soon developed this into a card game, which she played only amongst her friends and family before deciding to make a business out of it. SET was subsequently put on the market in 1990 [SET Enterprises Inc., 2013].

1.1.2 Rules

The rules of SET are quite simple. Each card in SET has one of three colours, numbers, shadings and shapes associated with them. There are 81 cards in total, and each has a unique combination of the traits described above.

[1]SET® is a registered trademark of SET Enterprises Inc., and SET game play is protected intellectual property.

In order to make a SET, we must choose three SET cards such that the four traits (colour, number, shape and shading) are either all the same or all different across the three cards. A game of SET consists of laying out twelve cards and repeatedly finding sets of three cards and covering these with new cards. The game ends when no more sets can be found and the deck of cards is exhausted.

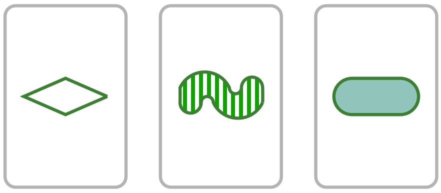

Figure 1: This group of cards would be considered a SET.

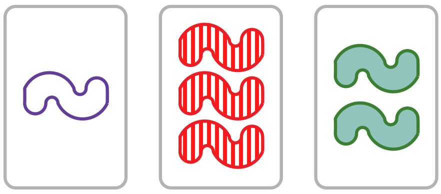

Figure 2: However, this group of cards would not.

1.2 Isomorphism

Soon after SET was released, mathematicians took an interest in the game and realised it can be modelled as AG$(4, 3)$, the four dimensional affine space over $\mathbb{Z}_3$ [Polster, 1998]. Each of the four traits of SET cards can be assigned to a dimension of the affine space. We have that the SET cards can be thought of as the points of AG$(4, 3)$. Even the object of the game, finding SETs of three cards with traits all the same or all different, is actually equivalent to finding lines of AG$(4, 3)$. Subsequently, papers were released by a range of authors exploring properties such as maximal caps (explored later in this report) and possible end-game scenarios [Davis et al., 2003; Do, 2005].

However, not much work has been done on exploring structures which are known to be embedded in AG$(4, 3)$ in the context of SET. Therefore, this report is a survey on some of the different sorts of objects which can be embedded in AG$(4, 3)$ and what their equivalent structures are in SET.

Using SET makes it much easier to visualise the four dimensional affine space we are working with and it enables us to more easily recognise patterns and nice symmetries.

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Throughout this report, we will use the convention that uppercase 'SET' will be used when referring to the card game and the groups of three cards that meet the requirements of SET mentioned above. On the other hand, 'set' in lowercase will be used to denote the mathematical concept of a set.

2 Generalised Quadrangles

The first objects that we explore in the context of SET are *generalised quadrangles*. Generalised quadrangles are well known structures in finite geometry and were introduced by Tits in 1959.

Definition 2.1 *A generalised quadrangle of order* (s, t) *is an incidence structure of points and lines such that:*

1. *Any two points are incident with at most one line.*
2. *Every point is incident with* $t + 1$ *lines.*
3. *Every line is incident with* $s + 1$ *points.*
4. *For any point* P *and line* ℓ *that are not incident, there is a unique point on* ℓ *collinear with* P*.*

In a landmark paper published in 1974, Buekenhout and Lefèvre completely classified all generalised quadrangles that can be embedded in finite projective spaces [Buekenhout and Lefèvre, 1974].

This motivated Thas to publish a paper in 1978 which completely classified generalised quadrangles in finite affine spaces [Thas, 1978]. Thas proved that there are three non-trivial generalised quadrangles that can be embedded in $\mathrm{AG}(4, 3)$. In this report, we focus solely on the unique generalised quadrangles $\mathrm{GQ}(2, 4)$ and $\mathrm{GQ}(2, 2)$, because of their other interesting properties when it comes to embedding other structures in $\mathrm{AG}(4, 3)$. Using Thas' construction of $\mathrm{GQ}(2, 4)$, we were able to generate coordinates for the points of the generalised quadrangle. We subsequently assigned one of the four traits of SET cards to each of the four varying coordinates and found a combination of SET cards that corresponded to the points of the generalised quadrangle. We used a similar method to determine the lines. Using this construction also determined the points and lines of $\mathrm{GQ}(2, 2)$ because it can be embedded in $\mathrm{GQ}(2, 4)$ [Payne and Thas, 2009].

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Note that these generalised quadrangles are unique up to equivalence.We have defined a particular correspondence between points and lines of AG(4, 3) and the 81 SET cards. By choosing a different correspondence (e.g. swapping the traits each coordinate correspond to), we may obtain generalised quadrangles represented by different SET cards.

Here, we have chosen one of these representations and it can be seen below.

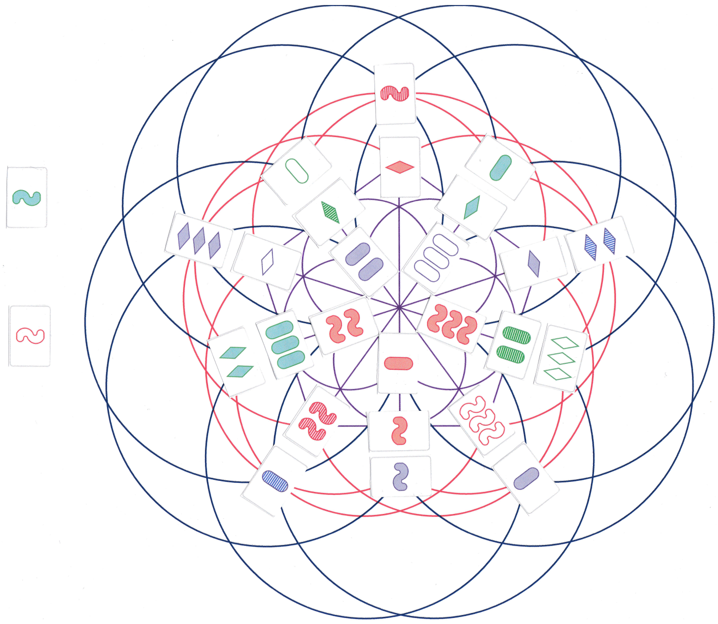

Figure 3: A representation of GQ(2, 4) in SET. [Polster, 1998]

Notice that we cannot nicely draw GQ(2, 4) in a planar fashion. Thus, we are left with two points (on the left in Figure 3) that appear to be on no lines. In fact, the top point is collinear with points on edges of the 'pentagon' in the generalised quadrangle and the points just outside these edges. The bottom non-planar point is collinear to the points outside the corners of the 'pentagon' and the opposite pentagon edge point.

Using SET as a way of visualising GQ(2, 4) enables us to see that if we draw a vertical line of symmetry down the centre of the generalised quadrangle, both the shapes and colours of the SET cards are symmetric. This would be much harder to

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discover if we were not using such a nice visualisation method.

3 Relative Hemisystems

Relative hemisystems are a very new concept in finite geometry, defined in 2011 by Penttila and Williford as an extension of hemisystems.

3.1 Hemisystems

Hemisystems themselves were first introduced by Segre in a landmark paper published in 1965 [Segre, 1965].

Definition 3.1 *A hemisystem of points of a finite geometry is a collection of points R such that R contains half the points on every line in the geometry.*

It is well known that each line in a finite geometry contains $q + 1$ points, where q is a parameter of the geometry, usually the size of the finite field the geometry is constructed over. As a result, in order to have a concept of exactly half the points on a line, q must be odd.

Segre constructed an example of a hemisystem on $Q^{-}(5, 3)$ in his paper using a Hermitian form [Segre, 1965]. Following Segre's paper, many mathematicians took a large interest in hemisystems because they give rise to a variety of other interesting mathematical structures such as partial quadrangles, strongly regular graphs and association schemes. However, all of the hemisystems found in subsequent years were shown to be isomorphic to Segre's. Interest soon waned in the subject, as mathematicians failed to find any new hemisystems to study. Thas even conjectured in 1995 that no hemisystem could be found on $Q^{-}(5, q)$ for $q \neq 3$ [Thas, 1995].

Then, in 2005, forty years after Segre's original paper, Penttila and Cossidente discovered an infinite family of hemisystems on $Q^{-}(5, q)$, for all odd $q \geq 3$ [Cossidente and Penttila, 2005]. This new infinite family resulted in the discovery of new partial quadrangles, strongly regular graphs and other structures. A large number of papers about hemisystems have been written ever since, proving the existence of even more infinite families.

3.2 Relative Hemisystems

In a 2011 paper, Penttila and Williford defined relative hemisystems using generalised quadrangles [Penttila and Williford, 2011]. Their motivation was to try and make the concept of hemisystems work for even values of q. A *relative hemisystem* may be defined as follows.

Definition 3.2 *Let S be a generalised quadrangle of order (q, q^2) containing a quadrangle S' of order (q, q). We call a subset H of the points disjoint from S' a relative hemisystem of S with respect to S' provided that for each line ℓ not completely contained in S' exactly half of the points on ℓ in $S \setminus S'$ lie in H. Also note that q must be even.*

Now we can consider the GQ$(2, 4)$ embedded in AG$(4, 3)$ that we found earlier. Recall that GQ$(2, 2)$ can also be embedded in GQ$(2, 4)$. Therefore, we can find relative hemisystems in AG$(4, 3)$ and thus SET.

Theorem 3.3 *There is a single relative hemisystem of points contained in SET (up to equivalence).*

Proof There are twelve points we can choose from initially to construct a relative hemisystem of points.

Suppose we choose a point outside a corner of the pentagon that forms part of GQ$(2, 2)$. Then since each line in $GQ(2, 4) \setminus GQ(2, 2)$ can only have one point in a relative hemisystem of points on it, this means we cannot choose any of the points outside the GQ$(2, 2)$ pentagon edges because they are adjacent to our original point. Nor can we choose the non-planar point collinear to all the points of the GQ$(2, 2)$ pentagon. This means we must choose all of the other points outside corners of the GQ$(2, 2)$ pentagon, which is consistent with the definition of a relative hemisystem of points because they are all non-adjacent. The only lines in $GQ(2, 4) \setminus GQ(2, 2)$ that are not covered now are containing the points adjacent to edges of the GQ$(2, 2)$ pentagon and the other non-planar point. Therefore, we are forced to choose the other non-planar point to complete our relative hemisystem.

Alternatively, suppose we initially choose a point adjacent to an edge of the GQ$(2, 2)$ pentagon. Then, we cannot choose any of the points adjacent to corners of the GQ$(2, 2)$ pentagon because our initial point is adjacent to all of them. We also cannot choose the non-planar point adjacent to that edge. Similar to the last case, in order for all of the lines to be covered, we must choose all of the remaining points adjacent to the GQ$(2, 2)$ pentagon edges and the other non-planar point for all of the lines to be covered exactly once.

Therefore, there are two relative hemisystem of points.

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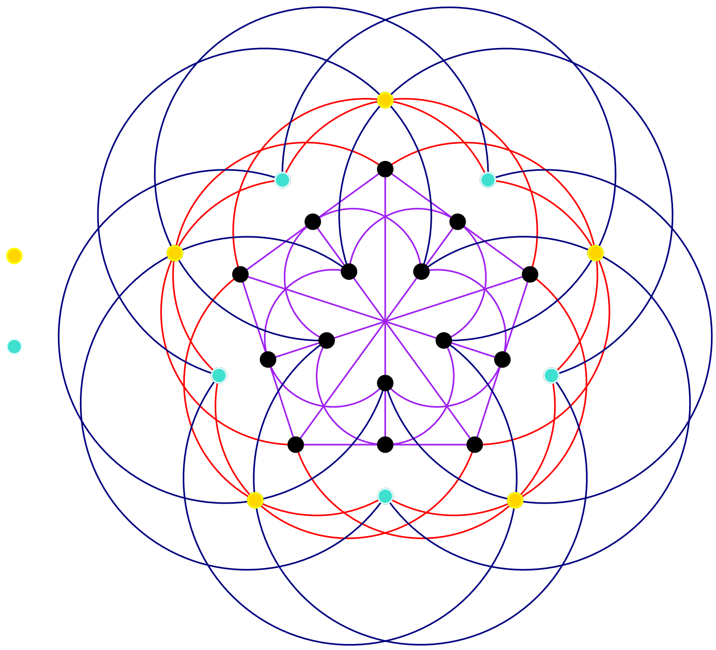
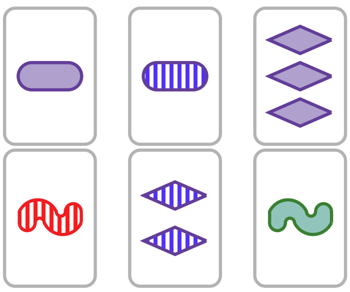


Figure 4: The two relative hemisystems of points.

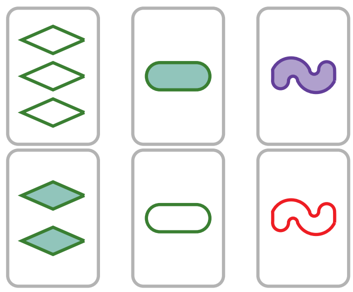
However, we can show they are equivalent. We can show there is a collineation (a bijection between points which preserves adjacency) between the two relative hemisystem of points by mapping each point in $GQ(2,4) \setminus GQ(2,2)$ to its 'opposite' point, that is, the point directly opposite it. This also means that the two collinear points are mapped to each other to preserve adjacency.

Therefore, there is one relative hemisystem of points up to equivalence.

We can see the representation of these two relative hemisystems in SET below.



(a) First Relative Hemisystem in SET.



(b) Second Relative Hemisystem in SET.

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We notice some interesting characteristics about the two relative hemisystems represented in SET. Firstly, they are both composed of four cards of one colour and one card each of the other two colours. The four cards of the same colour have the same shapes between the two relative hemisytems. We also notice that the two cards of unique colours in the relative hemisystems also share two traits - they are the same shape and both only have one symbol on their cards. These many similarities between the two relative hemisystems also makes their equivalence more apparent.

4 Maximal Caps

Now we consider maximal caps in AG$(4, 3)$ and the generalised quadrangles contained within it.

Definition 4.1 *A cap is a collection of points such that no three are collinear.*

A *maximal cap* then, is the largest cap we can find in the geometry that is not contained within another cap.

Since SETs are isomorphic to lines in AG$(4, 3)$, we are interested in maximal caps because they tell us how many SET cards we can deal before a SET is guaranteed.

4.1 Pellegrino Cap

The size of the maximal cap in AG$(4, 3)$ was discovered by Pellegrino in 1971 [Pellegrino, 1971]. It is of size 20 and is named the Pellegrino cap in his honour. This cap has already been explored in the context of SET by various authors (See [Davis et al., 2003; Do, 2005]).

An example of a Pellegrino cap in SET is shown below. Again note that there are many such examples, depending on how the mapping from points in AG$(4, 3)$ to SET cards.

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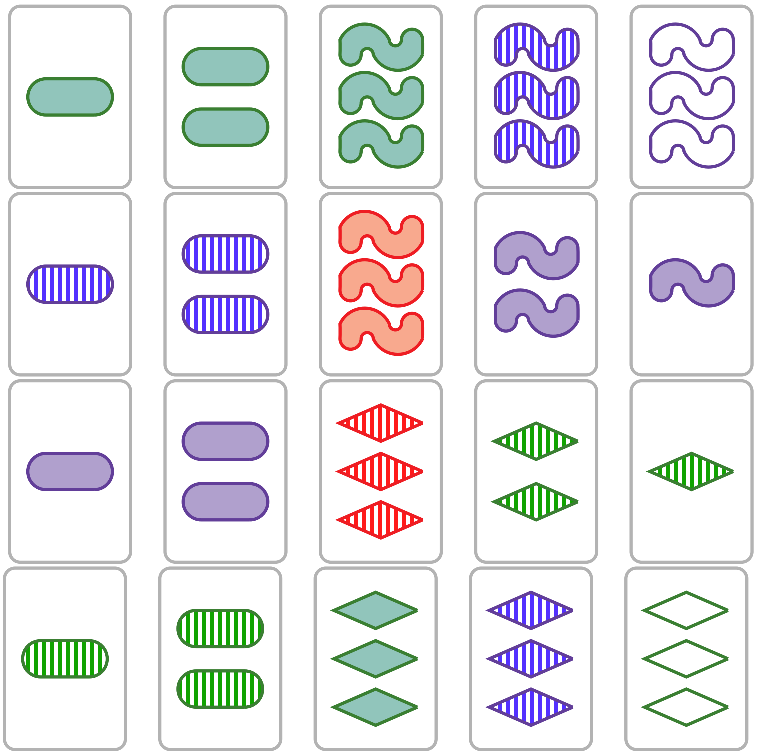

Figure 5: An example of a Pellegrino cap visualised in SET [Davis et al., 2003].

4.2 Maximal Cap in GQ(2,4)

Let us now look at maximal caps in GQ$(2, 4)$. As mentioned earlier, GQ$(2, 4)$ consists of 27 points and 45 lines. This is only a third of the points contained in AG$(4, 3)$, making the following result somewhat surprising.

Theorem 4.2 *A maximal cap in* GQ$(2, 4)$ *is of size 17.*

An example of such a cap for the GQ$(2, 4)$ represented in SET illustrated above is given below.

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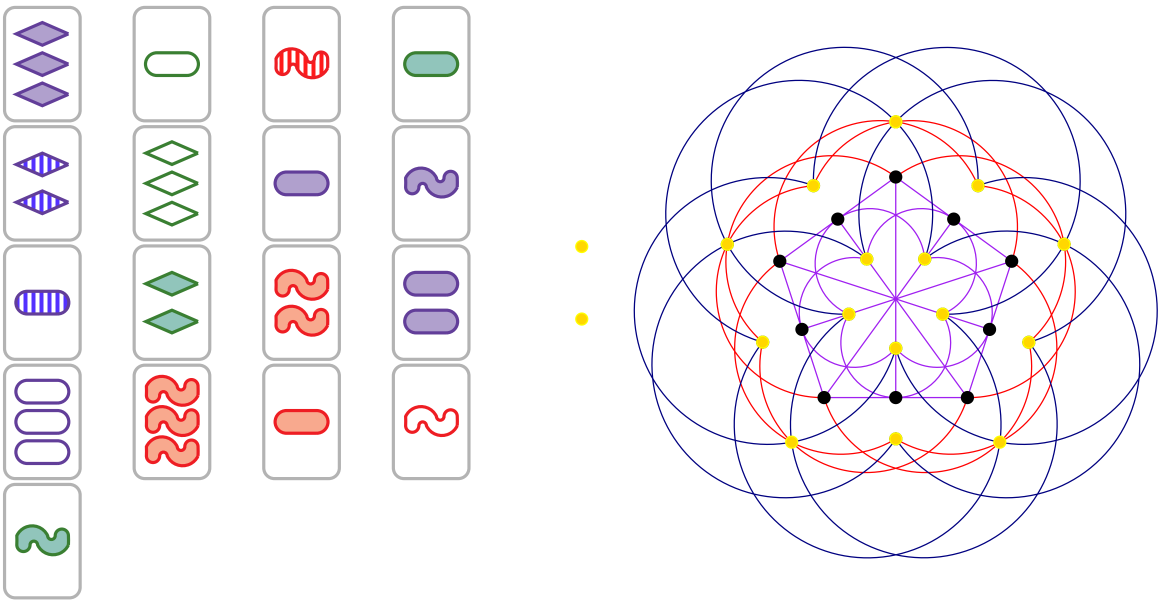

Figure 6: The SET cards corresponding to a maximal cap in GQ(2,4) and their corresponding positions on the illustration.

Notice that this collection of SET cards does actually contain a SET. It is still considered a cap of $GQ(2,4)$ because the line in $AG(4,3)$ corresponding to this SET does not lie in the generalised quadrangle.

We can see that the nice definition of a cap loses its meaning in the context of SET when we consider it over $GQ(2,4)$. However, it would be interesting to consider whether we can alter the rules of SET so that a cap in this setting regains meaning and becomes an important concept in the game.

5 Maximal Partial Ovoids

Related to maximal caps are maximal partial ovoids, which are another interesting concept to contemplate in terms of SET.

Definition 5.1 *A partial ovoid is a set of points, no two of which are collinear.*

Similar to a maximal cap, a *maximal partial ovoid* is the largest set of points we can find such that no two are collinear and the partial ovoid is not contained within another partial ovoid.

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5.1 Maximal Partial Ovoids in AG(4,3)

The concept of a partial ovoid is trivial in $AG(4, 3)$ because every two points are collinear. This is obvious if we consider it in terms of SET. If we choose any two cards, based on whether each of the four traits are the same or different between the cards, we can choose a third card to form a SET.

Therefore, it is only useful for us to study maximal partial ovoids in $GQ(2, 4)$.

5.2 Maximal Partial Ovoids in GQ(2,4)

Maximal partial ovoids in $GQ(2, 4)$ are less trivial to consider. We can prove the following result computationally.

Theorem 5.2 *The largest partial ovoid in GQ(2,4) is of size six.*

It turns out that the two relative hemisystems we found earlier are also maximal partial ovoids – they each have six points, none of which are collinear in the generalised quadrangle.

In fact, there are many more partial ovoids of size six in $GQ(2, 4)$.

Theorem 5.3 *There are 72 maximal partial ovoids in* $GQ(2, 4)$ *and they are all equivalent.*

Recall that when we say the maximal partial ovoids are equivalent, we mean that there are collineations that map the maximal partial ovoids to each other.

In conclusion, by contemplating the embeddings of geometric structures in AG(4,3) in terms of SET, not only do we realise a much simpler way of visualising these structures, we also observe the beautiful symmetries that arise. In future, it would be interesting to consider whether the rules of SET can be modified so that the structures discussed in this report can be discovered from playing the game, just as the lines of AG(4,3) do in the traditional SET game.

Acknowledgements

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