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The Plateau Problem

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The problem of determining a minimal surface spanned by a specified closed Jordan curve – the so-called Plateau problem – is the mathematical equivalent of investigating the soap film bounded by some closed wire. With the aim of studying the existence of solutions to Plateau's problem and their properties, we began with an introduction to the calculus of variations. In the calculus of variations we seek to either maximize or minimize quantities of the form

$$J = \int_a^b f(x, y(x), y'(x))dx\,.$$

It is readily demonstrated that stationary points of J must satisfy the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0\,.$$

We reproduced the derivation of this result in any dimension, and applied it to the famous example of the area functional (see below) to determine the equation prescribing a minimal surface expressed as a graph over some subset of R^n, with various boundary conditions. We proceeded to calculate the second variation – necessary for introducing conditions for a minimum (not merely extremum) of a functional.

We subsequently examined the preliminary components of a proof of the existence of solutions to Plateau's problem. Central to such an analysis is the justification of the analysis of the Dirichlet integral

$$D_B(X) = \frac{1}{2}\int_B \left(|X_u|^2 + |X_v|^2\right)dudv$$

in preference to the less favourable area functional

$$A_B(X) = \int_B \left(|X_u \wedge X_v|\right)dudv$$

where X is a surface of a specific class parametrized on the closure of the unit disk B, and where X maps the circle ∂B onto the given Jordan curve. Further, we were able to see that at least one solution to the problem of minimizing D in a given class exists for every given rectifiable Jordan curve.

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See www.ice-em.org.au/students.html#scholarship.