INTERNATIONAL CENTRE OF EXCELLENCE FOR EDUCATION IN MATHEMATICS

Estimating the excess precession of the perihelion of Mercury

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The aim of this project was to explore a novel technique for the construction of geodesics in a curved geometry. Our plan was to compare the predictions of the new technique against those of traditional methods for the simple problem of estimating the excess precession of the perihelion of Mercury. Unfortunately our implementation of the traditional techniques took far more time than we had expected and we ran out of time before the new numerical technique could be investigated.

Our first task was to develop the underlying mathematics that describes a geodesic in a curved spacetime. Such paths, geodesics, are postulated in General Relativity to be the paths followed by point objects. To a very good approximation Mercury, in relation to the Sun, may be treated as a point particle. Furthermore, the spacetime geometry for the Sun is well described by the Schwarzschild metric, which in spherical polar coordinates is given by

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 - \frac{2m}{2m-r}dr^2 - (rd\theta)^2 - (r\sin\theta\, d\phi)^2$$

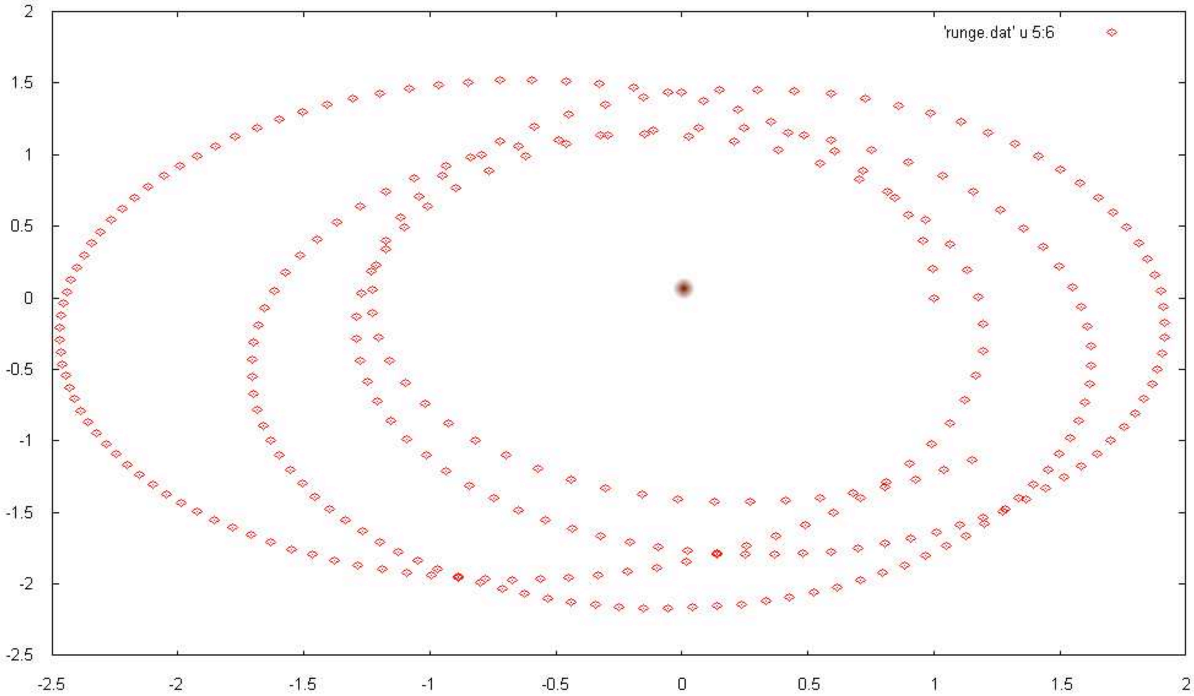
Thus our first task was to derive the geodesic equations for this spacetime. This we did using the Calculus of Variations which lead us to the following system of ordinary differential equations.

$$\begin{aligned}
\frac{d^2r}{d\tau^2} &= \frac{m}{r(r-2m)}\left(\frac{dr}{d\tau}\right)^2 - (r-2m)\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right) - \frac{m(r-2m)}{r^3}\left(\frac{dt}{d\tau}\right)^2 \\
\frac{d^2t}{d\tau^2} &= -\frac{2m}{r(r-2m)}\frac{dt}{d\tau}\frac{dr}{d\tau} \\
\frac{d^2\theta}{d\tau^2} &= \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 - \frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} \\
\frac{d^2\phi}{d\tau^2} &= -\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}
\end{aligned}$$

As our aim was to explore a novel numerical technique, we chose not to solve this system analytically (this can be done, but it is very tedious). Instead we prepared this system for traditional numerical integration (using a 4^{th} order Runge-Kutta method). This entails the introduction of 4 extra variables that play the role of the first time derivative. This leads to a system of eight first order differential equations. Since the geodesics are known to lie in a plane we made the further simplification of setting theta = pi/2 which reduced the system from 8 to 6 first order equations.

We solved this system numerically using the 4^{th} order Runge-Kutta method. The local truncation error of this method is of order [step size]^5.

A program was written in C defining each of the six first order ODEs, and six corresponding Runge-Kutta functions. The program produced a table of values for r, t and phi at each time step along the geodesic. We ran the program a number of times with various choices of the initial data, namely the values for all six variables. We also made provision for varying the mass of the central object. These resulting geodesics were then plotted using the graphics package Gnuplot.



This plot shows a very drastic precession of the perihelion of the ellipse per orbit. The orbit was found by experimenting with the central mass and the initial data.

The orbit of Mercury was found to be approximately circular, with a barely detectable precession. Measuring the precession quantitatively can be achieved by describing the exact geometry of the ellipse that Mercury traces out in its orbit.

Finally a three dimensional movie was made with Visual Studio of an object orbiting a large central mass from the position and time coordinates output by the constructed Runge-Kutta program.

The result of this project was the creation of a program that will output a movie of an object orbiting a large central body. By experimenting with the initial data and the mass of the central body, one could observe the affect of these variables on orbit, eccentricity and orbiting velocity.

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See www.ice-em.org.au/students.html#scholarship.