Minimal Surfaces: Bernstein's Theorem and the Weierstrass Representation Theorem

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For my project I studied minimal surfaces, and in particular Bernstein's Theorem and the Weierstrass Representation Theorem. Bernstein's Theorem says that any minimal surface that can be written as a graph over the whole plane must itself be a plane; and the Weierstrass Representation Theorem relates holomorphic functions and techniques from complex analysis with minimal surfaces, and gives a way of constructing minimal surfaces.

Initially, I had little knowledge of what a minimal surface was, so my first task was to find out more about minimal surfaces. This involved looking at texts on the calculus of variations, as minimal surfaces are those which have minimum surface area among all surfaces which share the same boundary. Calculating the first variation of area, it is possible to deduce that a surface with a fixed boundary is area minimizing if and only if it has zero mean curvature, i.e. it is a minimal surface.

I then looked at Bernstein's Theorem, most of the texts that I looked at dealt exclusively with the case in three dimensions. I saw a number of proofs of the three dimensional case, the most elegant of which used Liouville's Theorem from complex analysis. Having seen the three dimensional case, the natural question is whether it is true in every dimension. After doing some more research, I found that the answer is no, but it is true up to the seventh dimension. In order to understand the proof for the higher dimensional case, I had to research Geometric Measure Theory, which was particularly difficult as I had not done a standard measure theory course. The proof of the higher dimensional case basically comes down to the fact that any minimal surface which can be written as a graph over the whole plane must be a cone, and the fact that there are no minimal cones with singularities up to the seventh dimension, and cones without singularities are planes.

Overall, I found the experience to be thoroughly rewarding, being able to research deeply into an area of mathematics and then present my findings at the Big Day In. The project gave me an insight into what working mathematicians do.

Lachlan received an AMSI Vacation Scholarship in December 2008
See: www.ice-em.org.au/vs09.htm