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Geodesics in General Relativity

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In the theory of General Relativity, space-time is modeled as a 4 dimensional manifold with a certain geometry, and particles and light under the influence of gravity travel along special trajectories called *geodesics.* Geodesics can be thought of as the straightest possible trajectories in the geometry, the length minimizing curves in the geometry, or the trajectories with no acceleration. For instance the geodesics of a plane are just straight lines, and geodesics on a sphere are great circles.

My project centered around investigating the natural question:

Given a space-time M with a geometry g, what are the geodesics?

If we have this information then we learn much about the space-time for then we know the trajectories of particles and light. Then, for example, we can derive experimental tests to see if the given space-time we have chosen is an accurate model, or if the theory of General Relativity itself is accurate.

The geodesics of a manifold satisfy a certain system of 2^{nd} order differential equations (DEs), which in general are quite difficult to solve.

I was primarily concerned with the geodesics in the *Schwarzchild, Kerr,* and *Reissner-Nordstrom* spacetimes. These spacetimes correspond to the geometry outside a spherically symmetric mass with, respectively: no charge and no rotation, no charge and nonzero rotation, nonzero charge and no rotation.

It turns out that in these space times the geodesic DEs can be simplified considerably due to the symmetries of the spacetime.

In basic Newtonian mechanics many questions about particle trajectories can be answered using conservation of energy and (angular) momentum, without solving the 2^{nd} order DE, $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$, for the explicit form of the trajectory. They can also be used to simplify the 2^{nd} order DEs of motion. For example, for the problem of a particle subject to the gravitational field of a star, conservation of angular momentum and energy is used to simplify the system of three 2^{nd} order DEs in three variables to a system of two 1^{st} order DEs in two variables which can then be integrated.

The notion of a conserved quantity applies to any system of DEs. To be more precise, a conserved quantity is a function of first derivatives of paths that are constant on trajectories (solutions to the DEs).

The important thing about conserved quantities is that if we have enough of them, we will always be able to make similar simplifications as above and integrate the system.

Thus having conserved quantities are very useful for solving geodesic equations and DEs in general. We knew the expression for energy and angular momentum in the conservation laws from basic physics. It is easy enough to verify that they are indeed constant on trajectories by differentiating them and using the chain rule. But how do we find conserved quantities for other systems?

For geodesic DEs, we are saved by *Noether's Theorem.* In this special case the theorem says that every smooth symmetry of the spacetime manifold corresponds to a conserved quantity on the geodesics. In fact the theorem tells us exactly how to construct the expression for the conserved quantity.

It is easy to show that the Schwarzchild (and Reissner Nordstrom) geometry are spherically symmetric and invariant under time translation. This gives us 2 independent symmetries; rotation about an axis and translating by time. By Noether's theorem this gives us 2 conserved quantities, and it turns out that these are enough conserved quanitites to integrate the system.

From here we can easily get information about trajectories such as whether they escape, crash or orbit. Quantitative predictions such as the amount of bending of light in a gravitational field, and the amount of precession of periapsis in an almost elliptical orbit, can be made too. I applied the techniques used to get this information in the Schwarzchild spacetime to get similar predictions in the Reissner-Norstrom (charged, nonrotating mass). Even though it is very similar, I did run into some difficulties adapting the techniques and it was satisfying to overcome these difficulties.

During the summer I also learnt about Kerr spacetime, which models a rotating black hole with no charge, we do not have as many symmetries as it is not spherically symmetric. This means that the conserved quantities arising from Noether's theorem are not sufficient to determine the geodesics. However, Carter [Ca1968] discovered a conserved quantity on Kerr geodesics. It was not clear where this constant came from though; the original proof was a mess of algebraic manipulation. A slightly more illuminating explanation was given by Walker and Penrose [WaPe1970], who showed that the Carter constant arose from a *Killing tensor,* which is a generalization of the algebraic properties of the symmetries used before in Noether's theorem.

However, this only shifted the mystery onto the origin of the Killing tensor associated to the Carter constant. There was no clear geometrical interpretation of the Killing tensor, and Killing tensors are rare – they do not exist on many other

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important manifolds. Hence the existence of a Killing tensor on the Kerr spacetime seemed to be an extremely lucky coincidence. But Prince and Crampin [PrCr1983] showed that there was a natural geometric interpration of the Killing tensor and that in fact there is a there is a way to unify all geodesic conserved quantities and show that they all arise from a special group of symmetries on a space containing the original manifold.

I found this summer project extremely rewarding. Not only did I learn a lot of maths, many research skills, mathematical and nonmathematical, were developed: differential geometry, ability to read mathematical papers, prioritizing things to learn/do (unlike classwork, it is impossible to cover everything), using a CAS, preparing and giving a talk.

The Big Day In was also a great success as it allowed me to practice my presentation skills, meet many fellow maths students and diversify my mathematical view by listening to their well prepared talks across a wide range of topics.

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References

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