Smoothing Shock with Higher-order Parabolic PDEs

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Under the guidance of Dr Scott McCue and Dr Dann Mallet, solutions to first order hyperbolic and higher-order parabolic partial differential equations (PDEs) were investigated. The primary aim of the project was to study solutions to higher-order nonlinear parabolic PDEs with small amounts of damping, and compare these to solutions obtained when approximating the problem using a first-order hyperbolic model (with no damping at all).

For the purposes of this project, Burgers equation was investigated. This particular second-order nonlinear parabolic PDE was chosen since it is used to model many physical situations, and is a very special case which can be solved *exactly*. This property of Burgers equation enabled us to plot exact solutions without resorting to numerical schemes. The inviscid Burgers equation, which is a first-order nonlinear hyperbolic PDE found by setting the viscosity in Burgers equation to be zero, was also considered. Since the inviscid equation is hyperbolic, it can be solved exactly using method of characteristics which, for bell shaped initial data, leads to a multivalued solution.

This unphysical solution was then modified by the inclusion of a shock wave, and the location of the shock was solved numerically using a conservation of mass argument. The full Burgers equation was solved exactly using the Cole-Hopf transformation and then evaluated using Maple.

This solution is single valued for all time, and thus does not require any further assumptions involving shocks. To reconcile the two approaches, an alternate method for solving for the shock location and speed was then employed, using matched asymptotic expansions to find the interior solution of the full equation in the limit that the viscosity vanishes.

When comparing the full Burgers equation to the inviscid one, it was found that there are many strengths and weaknesses of each approach. First order hyperbolic PDEs are relatively simple to solve, however they may lead to multi-valued solutions which are unphysical. One option to deal with this non-uniqueness is to introduce the notion of a shock wave; however, shocks produce discontinuities in the solution which are also

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unphysical, and the jump conditions used to derive the speed and location of the shock are somewhat arbitrary. Using our example, we found that in the limit that viscosity vanishes, the solution to the full Burgers equation approaches the shock solution of the inviscid equation, the implication being that, at least for this example, the shock solution is a sufficient approximation to the higher-order problem.

The weaknesses of the first-order methods can also be overcome by including the higher-order terms in the full Burgers equation, however obtaining the solution to these problems can be much more difficult. Shocks are *regularised* by higher-order terms, with both the exact and interior layer solution being smooth and physically realistic. The higher-order terms provide the necessary physics of the problem, and never lead to discontinuous or multi-valued solutions. Thus overall, there was not a clearly superior method.

The Big Day In enabled me to meet students of all disciplines, and exposed me to presentations of many areas of mathematics and science. The experience and confidence gained from researching and presenting my findings in an environment with students and professionals was something I would not have gained in an undergraduate degree. I would recommend this summer scholarship to fellow students interested in mathematics, and would like to thank AMSI for giving me this opportunity to gain vital skills for future research and employment.

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