Cross Ratios and Thurston's Gluing Equations over Rings

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1. Introduction

We wish to glue ideal hyperbolic tetrahedra to form hyperbolic 3-manifolds. Ideal tetrahedra are tetrahedra which have their vertices at infinity (and so their vertices are not actually contained within the hyperbolic space). We could also glue typical tetrahedra to form 3-manifolds, so why do we choose to glue ideal tetrahedra? We do so because ideal tetrahedra are easier to parametrize. Not all gluings will result in the formation of hyperbolic 3-manifolds, the conditions required for the resultant space to be a 3-manifold are stated in terms of the tetrahedra which are to be glued; a nice parametrization allows for these conditions to be neatly given in terms of the parameters assigned to the tetrahedra. Another benefit of working with ideal tetrahedra is that when we glue tetrahedra, non-manifold points occur only along edges or at vertices, because the vertices of ideal tetrahedra do not actually belong to the tetrahedra we need only concern ourselves with what happens along edges. Because our specification for how we are to glue the tetrahedra will involve orientation, we give ourselves oriented, ideal hyperbolic tetrahedra.

2. Parametrising Ideal Hyperbolic Tetrahedra up to Oriented Congruence

We work in the upper half-space model of hyperbolic 3-space. This is the space $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ along with the metric given by $ds = \frac{dx^2+dy^2+dz^2}{z}$. We will be interested later in the orientation preserving isometry group of this space; this is the group $\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$, where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is identified with the Mobius transformation $z \mapsto \frac{az+b}{cz+d}$. An explanation of the isomorphism is in order. We identify the plane $z = 0$ with $\mathbb{C}$, and then as Figure [1](a) illustrates, because isometries map geodesics to geodesics, the action of isometries on geodesics is determined by the action on the endpoints at infinity. Extending this, Figure[1](b) illustrates that then, by considering two geodesics which intersect at that point, the action of isometries on any point in the hyperbolic 3-space is determined by their action on the endpoints of the geodesics at infinity, endpoints which lie in $\mathbb{C} \cup \{\infty\}$. The map $z \mapsto \frac{az+b}{cz+d}$ defines the action on the endpoints, where we make the usual definitions regarding the symbol ∞, such as $z/0 = \infty$ for non-zero z.

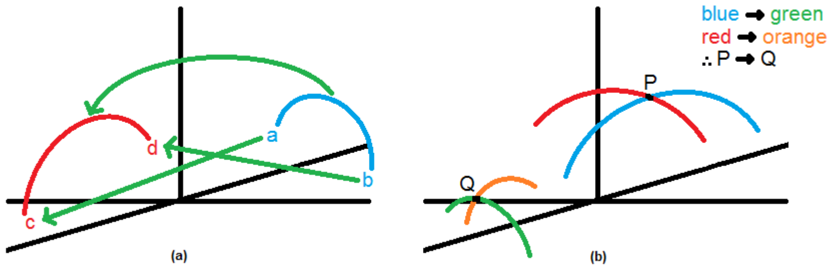

Figure 1: (a) The action of an isometry on geodesics is determined by the action on the endpoints at infinity (b) The action of an isometry on any point within the hyperbolic space is determined by it's action on geodesics

Having identified the plane at infinity, $z = 0$, with $\mathbb{C}$ we see that, in the upper half-space model, an ideal tetrahedron is specified by four pairwise distinct points in $\mathbb{C} \cup \{\infty\}$. Given an ideal tetrahedron, the link of each vertex is the set of geodesic rays in the tetrahedron passing through that vertex. These links can be concretely realised as the intersections of the tetrahedron with suitable horospheres about the vertices. See Figure [2].

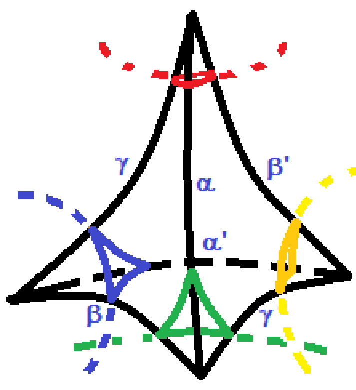

Figure [2]: The tetrahedron here is drawn abstractly, the vertices lie in $\mathbb{C} \cup \{\infty\}$; the broken lines indicate horospheres and the greek characters represent dihedral angles

Since the geometry of horospheres is Euclidean (which is easily seen in the upper half-space model for horospheres about ∞ by fixing z in $ds = \frac{dx^2+dy^2+dz^2}{z}$), all links are Euclidean triangles.

In particular, this means that the angles of each link, which are seen to be dihedral angles of the ideal tetrahedron, sum to π. Thus we have the following equations between the dihedral angles (corresponding to the red, blue, green and yellow links respectively):

$$\alpha + \beta' + \gamma = \pi$$

$$\alpha' + \beta + \gamma = \pi$$

$$\alpha + \beta + \gamma' = \pi$$

$$\alpha' + \beta' + \gamma' = \pi.$$

It is not hard to see that these equations imply that

$$\alpha = \alpha'$$

$$\beta = \beta'$$

$$\gamma = \gamma'.$$

In other words, in ideal tetrahedra opposite dihedral angles are equal.

Now, we take a moment to discuss orientation. In 2-space, be it Euclidean or hyperbolic, an orientation on a triangle is intuitively simple, and it can simply be seen as an order, up to cyclic permutation, on the three vertices of the triangle. Suppose our triangle has vertices v_1, v_2 and v_3. An orientation is an order on these three vertices; there are $3! = 6$ possible orderings, but up to cyclic permutation there are only $3!/3 = 2$. Thus there are two possible orientations on a triangle; pictorially these are the clockwise and counterclockwise orientations.

Moving up to three dimensions, what is an orientation on a tetrahedron? Again it is an order on the, in this case four, vertices v_1, v_2, v_3 and v_4. There are $4! = 24$ possible orderings of the vertices. In this case it isn't as clear whether or not orderings related via cylic permuations should be identified. We think in another direction. Given an ordering, it is intuitive to think that a transposition, that is, a permutation of the vertices which interchanges two vertices and leaves the remaining invariant, changes the orientation of the tetrahedron to an opposite form. As such any permutation which is a product of an even number of transpositions should be seen to preserve orientation and any which is a product of an odd number of transpositions to reverse the orientation. Thus the even permutations preserve the orientation, and an orientation on a tetrahedron is an ordering of the vertices up to even permutations. Note the analogy with the two dimensional case in that there the cyclic permutations correspond exactly to the even permutations. Once more there are $2 = 4!/12$ possible orientations; pictorially these are the left-handed and right-handed orientations.

Now, given an orientation on a tetrahedron, we may naturally infer an orientation on the links of its vertices. We do so by declaring that we view the links as described by the outward normal vectors from each vertex (that is, we view them from inside the tetrahedron), orienting their vertices in a clockwise manner. Our result that in ideal tetrahedra dihedral angles are equal then in particular implies that the links of each vertex are in the same oriented similarity class. See Figure [3].

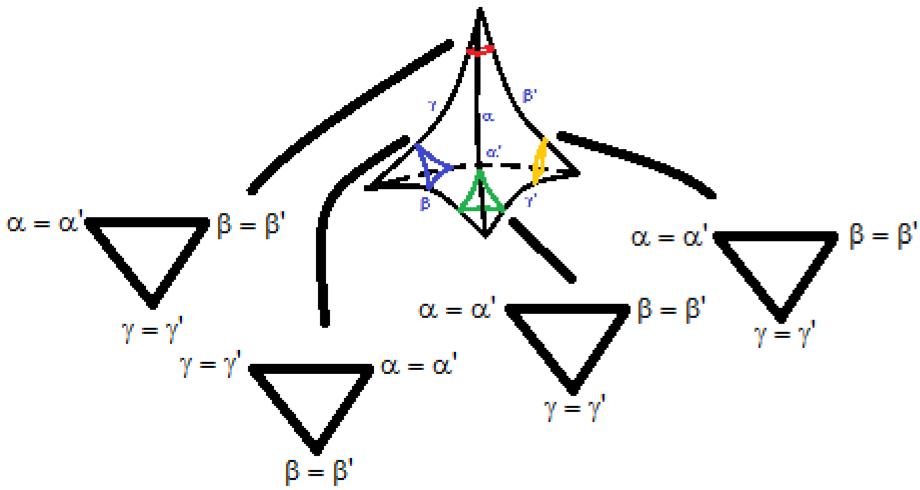

Figure [3]: The links of each vertex of an ideal tetrahedron are in the same oriented similarity class

It is clear that the oriented congruency class of an ideal tetrahedron determines the orientated similarity class of its links because two congruent ideal tetrahedra must have the same dihedral angles and because the two tetrahedra have the same orientation they will induce the same orientations upon the links of their vertices.

We assert the converse; that the oriented similarity class of the links of the vertices of an ideal tetrahedron determine it up to oriented congruence. To see this, suppose we have two ideal tetrahedra with links in the same oriented similarity class. Working in the upper half-space model, we can ensure both of our ideal tetrahedra have one vertex at ∞ via isometries. The triangles then determined by the vertices in the complex plane are in the same oriented similarity class. We can transform one tetrahedron into the other via a Euclidean similarity which preserves the complex plane. Such a similarity is an orientation preserving hyperbolic isometry.

Thus we see that parametrizing ideal tetrahedra up to oriented congruence is equivalent to parametrizing Euclidean triangles up to oriented similarity.

As such, we seek to parametrize Euclidean triangles up to oriented similarity. To do this, we regard $\mathbb{E}^2$ as $\mathbb{C}$. Given a triangle $\triangle(t, u, v)$, where the vertices t, u, v are oriented in a clockwise fashion, to the vertex v we associate the ratio

$$z(v) = \frac{t - v}{u - v}.$$

The number $z(v)$ is the point to which the third point t is mapped when v is sent to 0 and u to 1 via Euclidean orientation preserving similarities; namely translation, rotation and scaling. See Figure [4].

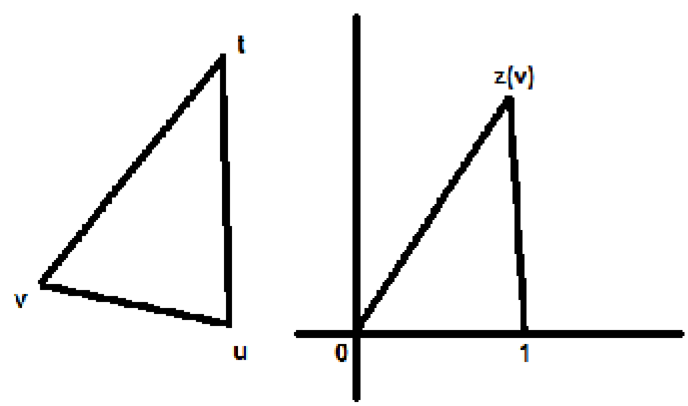

Figure [4]: Parametrizing Euclidean triangles up to oriented similarity

Similarly, the corresponding ratios for t and u are then

$$z(t) = \frac{u - t}{v - t}$$

$$z(u) = \frac{v - u}{t - u}.$$

We have the following identities

$$z(u) = \frac{1}{1 - z(v)}$$

$$z(t) = \frac{1}{1 - z(u)}.$$

It is seen from the above identities that any one of $z(t), z(u), z(v)$ determines the other two. Also, note that these identities are unchanged when we cyclically permute t, u, v.

It is clear that the similarity class of a Euclidean triangle determines the three values $z(v), z(t)$ and $z(u)$. Conversely, any one these values determine the similarity class of the triangle.

Given an oriented Euclidean triangle, $\triangle(t,u,v)$, we associate to it the triple $((t,z(t)),(u,z(u)),(v,z(v)))$, where the order of the entries is determined up to cyclic permuation and where the ordering is the same up to cylic permutation as the orientation of the triangle. In other words, we have that the following map parametrizes Euclidean triangles up to oriented similarity,

$$\{\text{Oriented Euclidean triangles} \to (\mathbb{C}^2)^3/\sim$$

$$\triangle(t,u,v) \mapsto ((t,z(t)),(u,z(u)),(v,z(v))),$$

where $\sim$ is the equivalence relation identifying those elements related by cyclic permuations and the order of the entries in the triple is the same, up to cyclic permutation, as the orientation of the triangle.

Practically this means that, to all oriented Euclidean triangles, to their vertices we associate the labels $z(t), z(u), z(v)$ and then we may speak of Euclidean triangles, up to oriented similarity, in terms of their associated parameters instead.

The parametrization of Euclidean triangles up to oriented similarity may be translated to a parametrization of ideal hyperbolic tetrahedra up to oriented congruence. To do this, given any ideal tetrahedron, we find the parametrization of the oriented similarity class of its links, and then we give the edges of the tetrahedron labels that were given to the corresponding vertices of the links. See Figure [5].

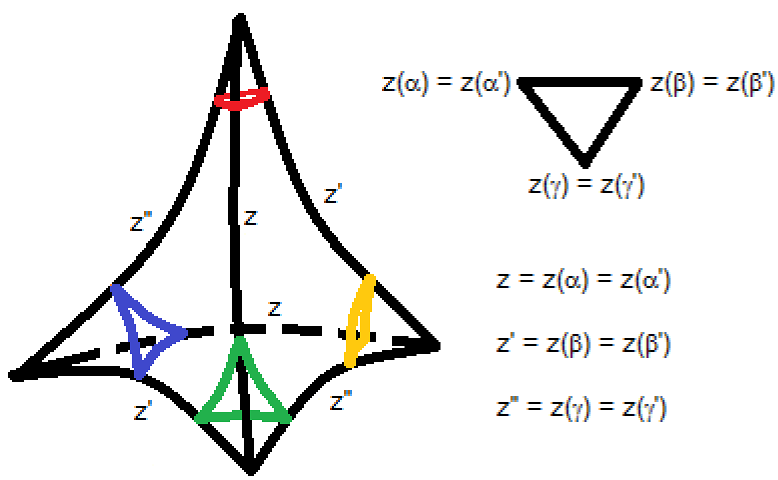

Figure [5]: Translating the parametrization of Euclidean triangles to ideal tetrahedra

As seen in Figure [5], this results in each edge being labelled by a complex number ($\neq 0, 1$), opposite edges being given the same label, and the three resulting labels, z, z' and z'' say, which we call the *shape parameters*, satisfying

$$z'(1-z) = z''(1-z') = z(1-z'') = 1.$$

We align the ordering of z, z', z'' with the orientation of the tetrahedron. These three values, the shape parameters, along with their ordering, then provide a parametrization of ideal hyperbolic tetrahedra up to oriented congruence in that the three shape parameters and their order determine the oriented congruency class of the ideal tetrahedron and conversely the oriented congruency class of an ideal tetrahedron determines the shape parameters that are to be associated to its edges, and an order on these parameters (that which is aligned with the orientation of the tetrahedron).

Working in the upper half-space model of $\mathbb{H}^3$, we wish to find the shape parameters of an ideal tetrahedron given its vertices. Suppose our tetrahedron has vertices $a, b, c, d \in \mathbb{C} \cup \{\infty\}$. Now, if any vertex, say d, is ∞, the shape parameters are easy to find because the link of this vertex will be similar to the triangle $\triangle(a, b, c)$ formed in the complex plane; and so the parameters will simply be $z(a), z(b)$ and $z(c)$, the order of them being determined by the orientation of the tetrahedron.

In our ideal tetrahedron with vertices at a, b, c, d, suppose that a, b, c are oriented in a clockwise manner (viewed from the vertex d). Via the orientation preserving hyperbolic isometry

$$z \mapsto \frac{\frac{\sqrt{b-d}}{\sqrt{a-d}\sqrt{b-a}}z - \frac{\sqrt{b-d}}{\sqrt{a-d}\sqrt{b-a}}a}{\frac{\sqrt{b-a}}{\sqrt{a-d}b-d}z - \frac{\sqrt{b-a}}{\sqrt{a-d}b-d}d} = \frac{b-d}{b-a}\frac{z-a}{z-d}$$

we map these vertices to $0, 1, \frac{(c-a)(b-d)}{(c-d)(b-a)}$ and ∞ respectively. Because isometries preserve shape parameters (that is to say, the congruency class of an ideal tetrahedron determines the oriented similarity class of its links), we see that the a, d edge is ascribed the label $\frac{(c-a)(b-d)}{(c-d)(b-a)}$; the other two labels then being

$$\frac{1}{1 - \frac{(c-a)(b-d)}{(c-d)(b-a)}} = \frac{(b-a)(c-d)}{(a-d)(b-c)}$$

for the c, d edge and

$$\frac{\frac{(c-a)(b-d)}{(c-d)(b-a)}}{\frac{(c-a)(b-d)}{(c-d)(b-a)} - 1} = \frac{(b-c)(a-d)}{(b-d)(a-c)}$$

for the b, d edge. Thus we see that the shape parameters of our ideal tetrahedra are in fact cross ratios of the vertices; the cross ratios being seen to reduce to the earlier labels by setting $d = \infty$.

In summary, we have a map

$$f : \{\text{Oriented ideal hyperbolic tetrahedra}\} \to (\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C})^3/\sim$$

$$\text{tetrahedron}(a, b, c, d) \mapsto (((a, b), (c, d), z_1), ((a, c), (b, d), z_2), ((a, d), (b, c), z_3)),$$

where $\sim$ is as before, the ordering of the entries of the triple follows the orientation of the tetrahedron; $z_{i+1}(1 - z_i) = 1$ where $z_4 = z_1$; and we have the explicit evaluation

$$z_1 = \frac{(c-a)(b-d)}{(c-d)(b-a)}$$

$$z_2 = \frac{(b-c)(a-d)}{(b-d)(a-c)}$$

$$z_3 = \frac{(b-a)(c-d)}{(a-d)(b-c)}.$$

3. Thurston's Gluing Equations

Now that we have parametrized ideal tetrahedra up to oriented congruence, we wish to glue them and note what conditions, in terms of the shape parameters, are required for the gluing to form a hyperbolic 3-manifold. Suppose a finite collection of ideal hyperbolic tetrahedra is glued to form a topological, non-compact 3-manifold M; we ask under which conditions on the tetrahedra does this give a (possibly incomplete) hyperbolic structure on M? As explained in the introduction, we concern ourselves only with what happens along the edges. First and foremost we label the edges of our tetrahedra following the parametrisation which we now possess. Consider an edge where n tetrahedra are to meet. Via an orientation preserving isometry we map one of the tetrahedra so that its vertices are at $0, 1, z_1, \infty$ (this is always possible because $\text{Isom}^+(\mathbb{H}^3)$ is triply transitive). What is z_1? Computing the appropriate cross ratio it is seen that z_1 must be the label of the $0, \infty$ edge — remembering that orientation preserving isometries preserve shape parameters, this is of course the same label as that of the preimage of this edge. See Figure [6](a). Next, we wish to map the tetrahedron glued to the preimage of the $0, z_1, \infty$ face to glue it to this $0, z_1, \infty$ face so that it's vertices are $0, z_1, w$ and ∞. What is w? Once again computing the relavent cross ratio, we find that w must be $z_1 z_2$ where z_2 is the label of the $0, \infty$ edge of the second tetrahedron. See Figure[6](b).

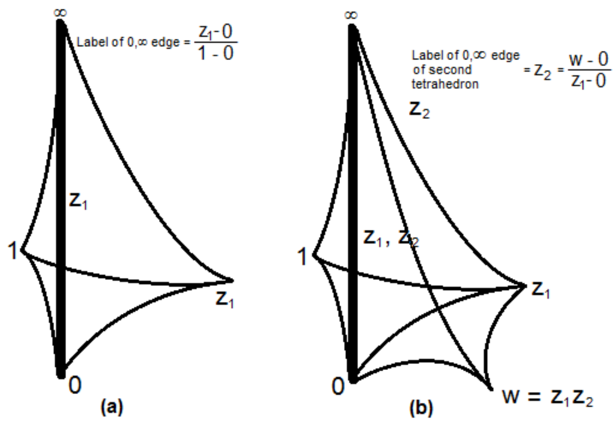

Figure [6]: Gluing tetrahedra around a common edge

Repeating this procedure we find that in the gluing, to achieve manifold points along the edge, we must have

$$z_1 \cdots z_n = 1,$$

where $z_1, \ldots, z_n$ are the labels of all the separate edges glued to form one common edge. Note that because orientation preserving isometries preserve shape parameters, this equation is independent of "where" the gluing actually takes place; that is to say, here we have moved the tetrahedra to have convenient vertices, however because the movements were achieved via orientation preserving isometries, the shape parameters, only parametrizing up to oriented congruence, see no difference. The above equation, along with the equations relating the parameters of a single tetrahedron to each other, are collectively known as the *hyperbolic gluing equations*, or *Thurston's gluing equations*.

4. Generalising the Equations to Rings

Since the hyperbolic gluing equations have integer coefficients, we may generalise $\mathbb{C}$ to any ring with identity. Further we drop the hyperbolic geometry. Instead, we consider 3-manifolds with triangulations, and we label the tetrahedra provided by the triangulations using elements from the ring. Note that the tetrahedra provided by the triangulations will in fact be typical tetrahedra, not ideal tetrahedra. We wish to emulate the parametrisation of the previous section in the context of general rings with identity.

Given a compact, oriented 3-manifold M, a triangulation $\mathcal{T}$ of M comprises the following information:

- a disjoint union $X = \sqcup_i \sigma_i$ of oriented Euclidean tetrahedra σ_i, and
- a set of orientation reversing affine homeomorphisms Φ between pairs of codimension-1 faces in X such that no face is left unpaired and M is homeomorphic to the quotient space X/Φ.

We can speak of the simplices in M, which are defined to be quotients of the σ_i. Note that we do not require that these simplices are embedded in M; for example, all four vertices may be glued to form a single vertex. However, the interiors of these simplices are embedded. See Figure [7].

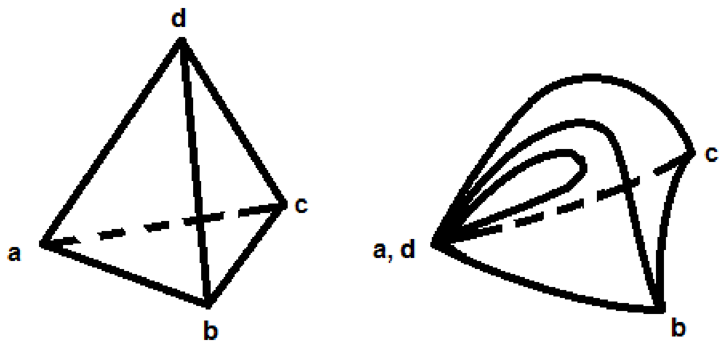

Figure [7]: Tetrahedra in the manifold may contain self-identification

Because some vertices may be identified, we will label *normal triangle types* instead of vertices. See Figure [8](a). Further, note that each opposite edge pair corresponds to a *normal quad type* as shown in Figure [8](b); we label these instead of edges. Given a triangulation, we label the set of normal triangle types and normal quadrilateral types by $\triangle$ and $\square$ respectively. An orientation on a tetrahedron can be seen as an ordering up to cyclic permutation on the normal quad types of the tetrahedron. If we have that the quad type q' follows the quad type q in this orientation, we write $q \to q'$.

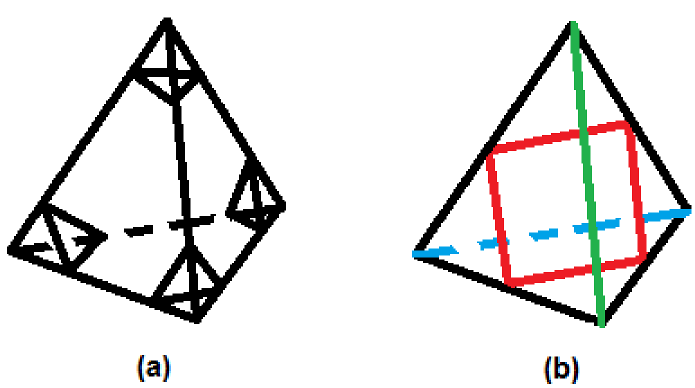

Figure [8]: (a) Normal triangle types; because the interiors of tetrahedra are still embedded, these cannot be lost under self-identification (b) Normal quad types; the red quad corresponds to the opposite to the pair of opposite edges, namely the green edge and the blue edge

Given a triangulated 3-manifold $(M, \mathcal{T})$ and a ring R with identity, a function $x : \square \to R$ is called a solution to Thurston's gluing equations associated to $\mathcal{T}$ if and only if

1. whenever $q \to q'$
$$x(q')(1 - x(q)) = (1 - x(q))x(q') = 1$$
2. for each edge e at which tetrahedra have been glued, if $q_1, \ldots, q_n$ are quads facing e labelled cyclically around e,
$$x(q_1) \cdots x(q_n) = x(q_n) \cdots x(q_1) = 1.$$

Note that the first condition implies that both $x(q)$ and $1 - x(q)$ are invertible elements. If the ring R is commutative, only one equation in both conditions is needed, and from now on it will be assumed that R is indeed commutative.

5. Solution by Cross Ratios

We construct here a solution to Thurston's gluing equations by emulating the shape parameters introduced earlier in the hyperbolic context. Earlier, the shape parameters were found to be cross ratios of the vertices, vertices which we may view to be in the complex projective line $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$. The natural generalisation of projective lines over rings is contained in the definition of the projective line, PR^1, over the ring R:

$$PR^1 = \{(a, b) \in R \times R : Ra + Rb = R\}/\!\sim$$

where $(a, b) \sim (u, v) \Leftrightarrow (a, b) = \lambda(u, v)$ for some unit $\lambda \in R$.

We look to label the vertices of our tetrahedra, provided by the triangulation of our manifold, with elements from PR^1. Following this we wish to take cross ratios of these vertices to give labels for our normal quad types. To see how we should define the cross ratio, we consider the case of the complex projective line $\mathbb{C}P^1$. Here we have a natural map $[z_1, z_2] \mapsto \frac{z_1}{z_2}$, where $z/0 = \infty$ for non-zero z. Using this we may compute the cross ratio of four points in $\mathbb{C}P^1$ as follows:

$$\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}; \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$$

$$\begin{aligned} &= \frac{(\frac{z_1}{z_2} - \frac{v_1}{v_2})(\frac{w_1}{w_2} - \frac{u_1}{u_2})}{(\frac{z_1}{z_2} - \frac{u_1}{u_2})(\frac{w_1}{w_2} - \frac{v_1}{v_2})} \\ &= \frac{\frac{z_1 v_2 - z_2 v_1}{z_2 v_2} \frac{w_1 u_2 - w_2 u_1}{w_2 u_2}}{\frac{z_1 u_2 - z_2 u_1}{z_2 u_2} \frac{w_1 v_2 - w_2 v_1}{w_2 v_2}} \\ &= \frac{(z_1 v_2 - z_2 v_1)(w_1 u_2 - w_2 u_1)}{(z_1 u_2 - z_2 u_1)(w_1 v_2 - w_2 v_1)}. \end{aligned}$$

Analogizing, we define the cross ratio of points $[(a_1, a_2)], [(b_1, b_2)], [(c_1, c_2)], [(d_1, d_2)] \in PR^1$ (in that order) as follows:

$$\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}; \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right) = \begin{bmatrix} (a_1 d_2 - a_2 d_1)(b_1 c_2 - b_2 c_1) \\ (a_1 c_2 - a_2 c_1)(b_1 d_2 - b_2 d_1) \end{bmatrix} \in R^2/\sim.$$

Note that the cross ratio is not necessarily in PR^1. It is simple to check that the cross ratio is well defined in the sense that replacing any of the points four points with $\sim$-equivalent points does not alter the value of the cross ratio. In the case that $(a_1c_2 - a_2c_1)(b_1d_2 - b_2d_1)$ is a unit in R we naturally associate the cross ratio with the quotient $\frac{(a_1d_2-a_2d_1)(b_1c_2-b_2c_1)}{(a_1c_2-a_2c_1)(b_1d_2-b_2d_1)}$.

To allow for a more concise presentation, we introduce the following notation. Given $(a, b), (c, d) \in R^2$ we define $<(a, b), (c, d)> = ad - bc$, the determinant of the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. With this notation, we have that, given $[A_1], [A_2], [A_3], [A_4] \in PR^1$ the cross ratio $([A_1], [A_2]; [A_3], [A_4])$ is the vector $\left[\begin{pmatrix} R_{1423} \\ R_{1324} \end{pmatrix} \right] \in R^2/\sim$, where $R_{ijkl} = <A_i, A_j><A_k, A_l>$.

Now, given any function $f : \triangle \to PR^1$ so that $f(t) = f(t')$ whenever two normal triangle types t, t' are adjacent (see Figure [9] below to see what we mean by adjacent), we define $F : \square \to R^2/\sim$ by $F(q) = (f(t_1), f(t_2); f(t_3), f(t_4)) = [z(q), y(q)]$ where t_1, t_2, t_3, t_4 are the four normal triangle types in the tetrahedron σ containing q so that q separates $\{t_1, t_2\}$ from $\{t_3, t_4\}$ and $t_1 \to t_2 \to t_3 \to t_4$ defines the orientation of σ.

Figure [9]: Adjacent normal triangle types

Note that $y(q) = -z(q')$, where $q \to q'$; this may be verified using the definition of the cross ratio. Now, as in the case when our ring is $\mathbb{C}$ we have that, in the case that $y(q)$ is a unit for all q, $x(q) = z(q)/y(q)$ is a solution to Thurston's gluing equations. This is a consequence of the following facts.

Given $[A_1], \ldots, [A_n] \in PR^1$, we have the following facts.

1. $([A_1], [A_2]; [A_3], [A_4]) + ([A_1], [A_3]; [A_4], [A_2]) + ([A_1], [A_4]; [A_2], [A_3]) = 0$

2. Given $[B], [C] \in PR^1$ and setting $[A_{n+1}] = [A_1]$,

$$\prod_{i=1}^{n}(< [B], [A_{i+1}] >< [C], [A_i] >) = \prod_{i=1}^{n}(< [B], [A_i] >< [C], [A_{i+1}] >).$$

Proof. Both facts follow systematically by setting $[A_i] = [(a_i, b_i)]$, $B = [(c, d)]$ and $C = [(e, f)]$ and evaluating the expressions which appear. □

Figure [10] below illustrates the motivation of the above properties.

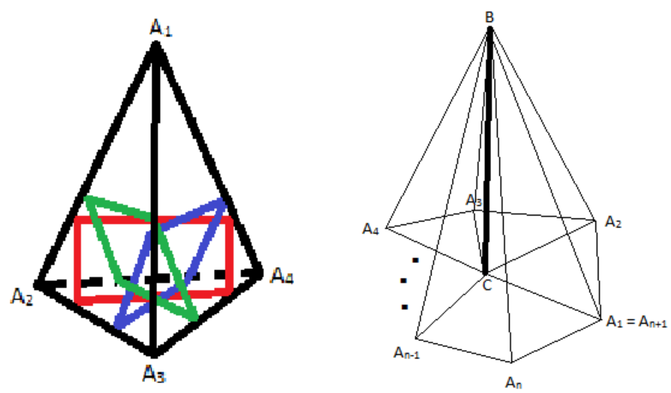

Figure [10]: Pictorial representation of properties of the cross ratio

As mentioned earlier, this solution to the gluing equation holds when $y(q)$ is a unit for all q. If in our triangulation $\mathcal{T}$ there is no self-gluing, and our ring is either a field or a ring with a group of units of cardinality larger than the number of non-adjacent normal triangle types, we may arbitrarily label the normal triangle types with distinct elements from the field or group of units and then this solution will be possible. If however, there is self-gluing, this solution breaks down as two normal triangle types in the same tetrahedron will be adjacent, must then receive the same label and so $y(q)$ will be zero for the corresponding normal quad type q. In this case, when self-gluing is present, we do not yet have any guide on how to solve the gluing equations. The next section shows that the existence of a solution gives us important information about our manifold.

6. Main Result

In the generalised context, it turns out that given that a solution to Thurston's equations exists at all in any commutative ring R with identity, we can ascertain information about the manifold. In particular, we have the following.

Main result [Luo]. *Suppose $(M, \mathcal{T})$ is an oriented connected closed 3-manifold* *with a triangulation $\mathcal{T}$ and R is a commutative ring with identity. If Thurston's equations on $(M, \mathcal{T})$ is solvable in R and $\mathcal{T}$ contains an edge which is a loop, then there exists a homomorphism from $\pi_1(M)$ to $PSL(2, R)$ sending the loop to a non-identity element. In particular, M is not simply connected.*

In this section we will work towards a proof of this fact.

Let $\pi : \widetilde{M} \to M$ be the universal cover and $\widetilde{\mathcal{T}}$ be the pull back of the triangulation $\mathcal{T}$ of M to $\widetilde{M}$. Denote by $\widetilde{\triangle}$ and $\widetilde{\square}$ the sets of all normal triangle types and quad types in $\widetilde{\mathcal{T}}$ respectively. The covering map induces a surjection π_* from $\widetilde{\triangle}$ and $\widetilde{\square}$ to $\triangle$ and $\square$ respectively so that $\pi_*(d_1) = \pi_*(d_2)$ if and only if d_1 and d_2 differ by a deck transformation element.

Suppose that $x : \square \to R$ is a solution to Thurston's gluing equations on $\mathcal{T}$. Define $z : \square \to R^*$ by $z(q) = x(q), z(q') = -1$ and $z(q'') = 1 - x(q)$ where $q \to q' \to q''$ (R^* is the group of units of the ring R). We make these definitions so that we have $x(q) = -z(q)/z(q')$; and so we set $w : \square \to PR^1$ to be the map $w(q) = [z(q), -z(q')]$ and further, we let $\widetilde{w} = w\pi_*$ be the associated map on $\widetilde{\square}$.

Given a solution x to Thurston's gluing equations on $(M, \mathcal{T})$, we call a map $\phi : \widetilde{\triangle} \to PR^1$ a *pseudo developing map* associated to x if

1. whenever t_1, t_2 are two normal triangles in $\widetilde{\triangle}$ are adjacent, then $\phi(t_1) = \phi(t_2)$,

2. if t_1, t_2, t_3, t_4 are four normal triangles in a tetrahedron σ then $<A_j, A_k>$ is a unit for distinct $k, j = 1, 2, 3, 4$ where $\phi(t_i) = [A_i]$ and

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \widetilde{w}(q)$$

   where $t_1 \to t_2 \to t_3 \to t_4$ determines the orientation of the tetrahedron σ and q is a normal quad type in σ separating $\{t_1, t_2\}$ from $\{t_3, t_4\}$.

The idea here is that we wish to label the tetrahedra in the universal cover up above in a way that is in some sense consistent with our current labeling (the solution x) of the tetrahedra in the manifold down below. It can be proven that, given any solution x there exists a pseudo developing map associated to x. To show this, we require the following lemma.

Lemma 6.1 *Suppose t_1, t_2, t_3 and t_4 are the four normal triangle types in a tetrahedron σ so that $t_1 \to t_2 \to t_3 \to t_4$ determines the orientation of σ and q is a quad type in σ separating t_1, t_2 from t_3, t_4. If $\phi(t_i) = [A_i] \in PR^1$, $i = 1, 2, 3$ are defined and $<A_i, A_j>$ is a unit in R for distinct $i, j = 1, 2, 3$, then there exists a unique $\phi(t_4) = [A_4] \in PR^1$ so that $[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \widetilde{w}(q)$. Furthermore, the property that $<A_i, A_j>$ is a unit in R for distinct $i, j = 1, 2, 3$ extends to distinct $i, j = 1, 2, 3, 4$.*

Proof. (Follows the proof in [1]) We use the property that, for $Y \in GL(2,R)$, $(YA_1, YA_2; YA_3, YA_4) = det(Y)^2(A_1, A_2; A_3, A_4)$, which may be verified by direct calculation. Set $A_i = (a_i, b_i)$, $\widetilde{w}(q) = [(c_1, c_2)]$, and consider $X = \frac{1}{<A_1,A_2>}\begin{pmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{pmatrix} \in GL(2,R)$. Upon left multiplication, X maps A_1 and A_2 to $(1,0)$ and $(0,1)$ respectively. By the above property, after replacing A_i by XA_i we may assume that $A_1 = (1,0)$ and $A_2 = (0,1)$. Then computing the cross ratio we have $(A_1, A_2; A_3, A_4) = (-a_3b_4, -a_4b_3)$. By the assumption that $<A_i, A_3>$ is a unit for $i = 1,2$ we see that a_3 and b_3 are units. It follows that $[A_4]$ must uniquely be $[-c_1/b_3, -c_2/a_3]$. Then that $<A_i, A_j>$ is a unit in R for distinct $i, j = 1,2,3,4$ is easy to verify. □

Given the above lemma we construct our map by combinatorial continuation; applying the following procedure on connected components of the universal cover separately. We arbitrarily choose a tetrahedron and label three normal triangle types t_1, t_2 and t_3 by $[1,0]$, $[0,1]$ and $[1,1]$. Having done so, we label the fourth normal triangle type t_4 with the unique element of PR^1 such that $[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = \widetilde{w}(q)$ as described in the Lemma 6.1. Properties of the cross ratio then ensure that the two other quad types also receive the correct label. Now, given any tetrahedron glued to a face of our labelled tetrahedra, it will share three of its normal triangle types with the first tetrahedra so that these three normal triangle types must be labelled with the label of their respective adjacent normal triangle types. We then extend ϕ to the fourth normal triangle type of this second tetrahedron once again with our lemma above. Continuing in this fashion we will have labelled all the tetrahedra in the fashion we desire.

There is one issue which we did not address above. If a tetrahedron can be reach from the initial tetrahedron via multiple paths, it has not been shown that the resulting labels from both paths will coincide. This is a consequence of the gluing equations — a proof may be found in [1].

Having constructed the pseudo developing map, we find that we have the following.

Theorem 6.2 *Given a solution to Thurston's equations x and an associated pseudo developing map ϕ, there exists a homomorphism $\rho : \pi_1(M) \to PSL(2,R)$ such that for all $\gamma \in \pi_1(M)$, considered as a deck transformation,*

$$\phi(\gamma) = \rho(\gamma)\phi$$

as functions on the normal triangle types.

Proof. (Follows the proof in [1]) The proof requires the following lemma.

Lemma 6.3 *Suppose $A_1, \ldots, A_4, B_1, \ldots, B_4 \in R^2$ so that $<A_i, A_j>$ and $<B_i, B_j>$ are units for distinct $i, j = 1,2,3,4$ and $[A_1, A_2; A_3, A_4] = [B_1, B_2; B_3, B_2]$. Then there exists a unique $X \in PGL(2,R)$ so that $[XA_i] = [B_i]$ for all i.*

Proof. (Follows the proof in [1]) As in the proof of Lemma 6.1, we may assume $A_1 = B_1 = (1, 0)$ and $A_2 = B_2 = (0, 1)$. Then $[XA_i] = [B_i]$ for $i = 1, 2$ tells us that X must be diagonal. Then it can be seen that the matrix we require is $[X]$ where $X = \begin{pmatrix} c/a & 0 \\ 0 & d/b \end{pmatrix}$. Note that a and b must be units according to the conditions that $<A_i, A_j>$ and $<B_i, B_j>$ are units for distinct $i, j = 1, 2, 3, 4$. $\square$

Now, the construction of ρ is as follows. Fix an element $\gamma \in \pi_1(M)$. By the construction, $\pi_1(M)$ acts on $\widetilde{M}$, $\widetilde{\mathcal{T}}$, $\widetilde{\triangle}$ and $\widetilde{\square}$ so that $\pi_*(\gamma) = \pi_*$ for $\gamma \in \pi_1(M)$. This implies

$$[\phi(t_1), \phi(t_2); \phi(t_3), \phi(t_4)] = [\phi(\gamma t_1), \phi(\gamma t_2); \phi(\gamma t_3), \phi(\gamma t_4)]$$

for all normal triangles $t_1, \ldots, t_4$ in each tetrahedron σ in the triangulation. By Lemma 6.3, there exists an element $\rho_\sigma(\gamma) \in PSL(2, R)$ so that $\phi(\gamma t_i) = \rho\sigma(\gamma)\phi(t_i)$ where the t_i are in σ. We claim that $\rho\sigma(\gamma) = \rho\sigma'(\gamma)$ for any two σ, σ. Indeed, since any two tetrahedra can be joined by an edge path in the graph G, it suffices to show that $\rho\sigma(\gamma) = \rho\sigma'(\gamma)$ for two tetrahedra sharing a codimension-1 face τ. Let t_1, t_2, t_3 and t'_1, t'_2, t'_3 be the normal triangles in σ and σ' respectively so that t_i and t'_i are adjacent and and t_1, t_2, t_3 are adjacent to τ. Now $\phi(t_i) = \phi(t'_i)$ and $\gamma\phi(t_i) = \gamma\phi(t'_i)$, therefore, $\rho\sigma(\gamma)\phi(t_i) = \rho\sigma'(\gamma)\phi(t_i)$ for $i = 1, 2, 3$. By the uniqueness part of Lemma 6.3, it follows that $\rho\sigma(\gamma) = \rho\sigma'(\gamma)$. The common value is denoted by $\rho(\gamma)$. Given $\gamma_1, \gamma_2 \in \pi_1(M)$, by definition, $\rho(\gamma_1\gamma_2)\phi = \phi(\gamma_1\gamma_2) = \rho(\gamma_1)\phi(\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)\phi$ and the uniqueness part of Lemma 6.3, we see that $\rho(\gamma_1\gamma_2) = \rho(\gamma_1)\rho(\gamma_2)$, that is to say, ρ is a group homomorphism from $\pi_1(M)$ to $PSL(2, R)$. $\square$

With the homomorphism ρ we may finally prove our main result. This proof is reproduced from [1].

We first note that a loop in M will lift to arc in the universal cover joining different boundary components. For suppose otherwise that there exists an edge $e \in \mathcal{T}$ whose lift is an edge $e^* \in \widetilde{\mathcal{T}}$ joining the same boundary component of $\widetilde{M}$. Take a tetrahedron σ containing e^* as an edge and let t_1, t_2, t_3, t_4 be all normal triangles in σ so that t_1, t_2 are adjacent to e^*. By definition, the pseudo developing map $\phi : \widetilde{\triangle} \to PR^1$ satisfies the condition that $<A_j, A_k>$ is a unit for distinct $k, j = 1, 2, 3, 4$ where $\phi(t_i) = [A_i]$. In particular, $\phi(t_1) \neq \phi(t_2)$. On the other hand, since e^* ends at the same connected component of ∂M which is a union of adjacent normal triangles, there exists a sequence of normal triangles $s_1 = t_1, s_2, \ldots, s_n = t_2$ in $\widetilde{\triangle}$ so that s_i is adjacent to s_{i+1}. In particular, $\phi(s_i) = \phi(s_{i+1})$. This implies that $\phi(t_1) = \phi(t_2)$ contradicting the fact that $\phi(t_1) \neq \phi(t_2)$.

Now, suppose that e is an edge in $\mathcal{T}$ ending at the same vertex v in $\mathcal{T}$, let $\gamma \in \pi_1(M, v)$ be the deck transformation element corresponding to the loop

e. We claim that $\rho(\gamma) \neq id$ in $PSL(2, R)$. Indeed, suppose e^* is the lifting of e. Then by the statement just proved, e^* has two distinct vertices u_1 and u_2 in $\widetilde{M}$ and $\phi(u_1) \neq \phi(u_2)$. By definition $\gamma(u_1) = u_2$. It follows that $\phi(u_2) = \phi(\gamma u_1) = \rho(\gamma)\phi(u_1)$. Since $\phi(u_1) \neq \phi(u_2)$, we obtain $\rho(\gamma) \neq id$. This proves our main result.

7. Examples of Solving Thurston's Equations

Our result from the previous section motivates the study of the existence of solutions in rings R to Thurston's equations for triangulations of 3-manifolds. Here we provide some examples of solving the equations in some finite rings. These examples are taken from [1].

If $R = F_3$, the field with three elements, then since no quad may receive the labels 0 or 1 (because of the parameter relations), each edge must be labelled 2. At each edge the gluing equation then becomes $2^k = 1$ where k is the number of tetrahedra which meet at the edge. This is true if and only if k is even. Thus the gluing equations are solvable in F_3 if and only if each edge has even degree.

If $R = F_5 = \{0, 1, 2, 3, 4\}$, again no quad may receive the labels 0 or 1; in this case the label of each quad may be 2, 3 or 4. Since $1 = 2^4$, $2 = 2^1$, $3 = 2^3$ and $4 = 2^2$ each label must be 2^k for some $k \in \{1, 2, 3\}$. Because $1/(1-2) = 4 = 2^2$, $1/(1-3) = 2 = 2^1$ and $1/(1-4) = 3 = 2^3$, we see that there are three, and exactly three, ways to label the quads in each tetrahedron — by labelling using 2,4 and 3, in that order, following the orientation of the quads; there being three possible ways to do this. This gives a solution to the gluing equations if and only if at each edge, the powers of 2 taken from each label sum to a multiple of four.

If $R = F_{2^2} = \{0, 1, a, b\}$ where $b = a + 1 = a^2$ and $a^3 = 1$, we have that the quad labels may be a or b. Since $1/(1-a) = a$ and $1/(1-b) = b$, either all the quads in a tetrahedron are labelled a, or all are labelled $b = a^2$. Thus for us to have a solution to the gluing equations, we must have that at each edge, the sum of the number of tetrahedron with all labels a and twice the number of tetraheron with all labels b must be a multiple of 3.

These are simple examples which can be worked out by hand. Our direction of research from here is to use the computer software Regina and Singular to first provide us with triangulations of well-known 3-manifolds and then to study solution sets, over various rings, of the gluing equations corresponding to these triangulations. The aim is to develop general procedures, for particular rings, for determining whether or not a solution exists to the gluing equations.

References

[1] Luo, Feng; Solving Thurston Equation in a Commutative Ring; arXiv: 1201.2228, 2012.