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Roots of Chromatic Polynomials

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Graph theory is the study of discrete mathematical structures called graphs. Its results have many applications in areas such as computer science, economics and the natural sciences. One can go further and explore the colourings of a graph. This branch of graph theory is called Graph Colouring.

A proper colouring of a graph is an assignment of colours to the vertices of a graph such that no vertices sharing an edge receive the same colour. In 1912 Birkhoff [1] introduced the chromatic polynomial to provide a quantitative approach to the study of proper colourings. The chromatic polynomial of a graph G is the polynomial $P_G(q)$ which counts the number of proper colourings of the graph with *q* colours. The graph in *Figure 1* is called the Petersen graph and has a chromatic polynomial of the form:

$$P_G(q) = q(q-1)(q-2)(q^7 - 12q^6 + 67q^5 - 230q^4 + 529q^3 - 814q^2 + 775q - 352)$$

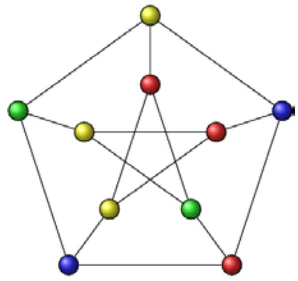

Figure 1. The Petersen Graph

Although originally conceived for the positive integers, a chromatic polynomial can be evaluated at both real and complex arguments. Both its real and complex zeros can also be studied and in fact the complex chromatic roots are of considerable interest to statistical physicists.

My research began with learning the basics of graph theory then progressing to the more relevant topic of graph colourings. I studied the theory of chromatic polynomials and surveyed the developments in the field before doing some active research of my own.

I learnt to compute the chromatic polynomial of various 'simple' graphs such as the tree and complete graphs and then explored techniques required to compute chromatic polynomials of relatively harder graphs. I also looked at recursive families of graphs, whose chromatic polynomials satisfy a linear recurrence. To compute the chromatic polynomial of such graphs required constructing an auxiliary equation from the graphs recursive relation and then solving a set of linear equations.

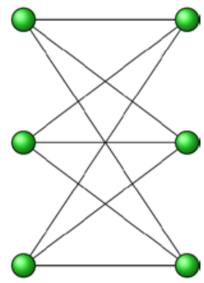

Figure 2. Complete bipartite graph $K_{3,3}$

I then looked at a special class of graphs called complete bipartite graphs whose vertices can be divided into two disjoint sets U and V, such that every vertex in U connects to every vertex in V. An example of a bipartite graph with 3 vertices in each disjoint set is shown in Figure 2. The technique used to compute the chromatic polynomial of such graphs essentially consisted of partitioning the vertices into various sized sets and summing over those partitions.

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From the knowledge of bipartite graphs I extended this logic to computing the chromatic polynomial of complete tripartite graphs. These graphs have a similar construction though the vertex sets can be divided into three disjoint sets.

The method of constructing the chromatic polynomial from this simple principle provided an elegant insight into the intrinsic structure of their polynomial. This assisted in the next part of investigation which was to answer a mysterious long standing question about chromatic polynomials.

The combinatorial interpretations for the multiplicity of 0 and 1 as chromatic roots were already well understood and so the question was to answer the following question: “Is there a combinatorial interpretation for the multiplicity of 2 as a root of the chromatic polynomial of a graph?" By looking at the structure of graphs and their corresponding chromatic polynomials the next logical step in the progression required that all 3-connected graphs had 0, 1, and 2 as simple chromatic roots. If this were the case then a logical argument could have been formed to answer the initial question posed.

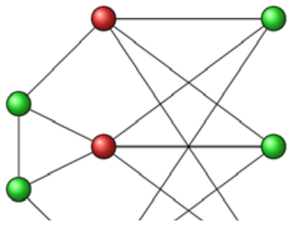

Figure 3. Graph with "multiplicity 2 for the root 2"

It was found that this was in fact not the case and that there existed 3-connected graphs for which 2 was a multiple chromatic root. One such graph is shown in Figure 3.
After careful investigation the vertices coloured in red were found to be the ones to blame for its chromatic polynomial having the aforementioned property.
In my research I hoped to gain a better understanding for the structure of graphs, which had this property. What I found was that an infinite family of such graphs could be constructed, thereby halting any hope that a simple structure were to blame. This meant that if there were ever a simple explanation to the initial question posed, that it was far less intuitive then once suspected.

Receiving an AMSI Vacation Scholarship was a great experience and allowed me to get an idea as to what mathematics research is all about. It provided the right amount of order and flexibility which I believe was paramount to getting the most out of the experience. The CSIRO Big Day In was a perfect way to finish the project and allowed students with a range of science backgrounds to share their enthusiasm and ideas.

I would like to thank AMSI for their generosity, in addition to CSIRO and the University of Western Australia for making this possible. A special thanks must go to my supervisor Dr. Gordon Royle whose patience and guidance were second to none.

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References

[1] G. D. Birkhoff, A determinant formula for the number of ways of colouring a map, Ann. of Math. (2) 14 (1912/1913), 42-46

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