Birman-Wenzl-Murakami Algebra and Non-Planar Logarithmic Models

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Abstract

The Birman-Wenzl-Murakami (BMW) algebra is an example of a braid-monoid or tangle algebra which was introduced in the context of knot theory to study 2-d projections of knots through over-crossings and under-crossings. It is a non-commutative finite algebra without a natural distinguished basis and, therefore, no simple canonical form for its independent words. Therefore, it is non-trivial to do general algebra in the BMW. The loop representation of the BMW algebra was obtained by implementing a computer system to do general algebra in the BMW. Physically, this is related to exactly solvable 2-d lattice models. By defining certain face operators in terms of the braid from the BMW, it can be shown that they satisfy the Yang-Baxter equation, implying that the model is related to a 2-d lattice model that is exactly solvable.

Contents

1 Introduction

The Birman-Wenzl-Murakami (BMW) algebra was introduced by Jun Murakami in 1987 [1] and separately by Joan S. Birman and Hans Wenzl in 1989 [2]. It is a braid-monoid algebra where the braid satisfies a cubic and the monoid is quadratic in the braids. This algebra reduces to the Brauer algebra under certain conditions. The Brauer algebra was first introduced by Richard Brauer in 1937 [3]. The Temperley-Lieb algebra, which was introduced by H. N. V. Temperley and E. H. Lieb in 1971 [4], is a sub-algebra of the BMW algebra.

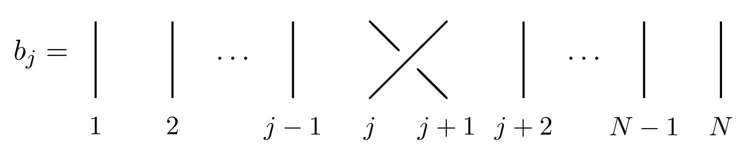
2 Braid-Monoid Algebras

A braid-monoid algebra is a non-commutative algebra generated by invertible braid operators b_j and Temperley-Lieb operators e_j with $j = 1, 2, \ldots, N-1$ satisfying

$$f(b_j) = 0, \qquad e_j = g(b_j) \tag{2.1}$$
$$b_j b_j^{-1} = I \tag{2.2}$$
$$b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1} \tag{2.3}$$
$$b_i b_j = b_j b_i \qquad |i-j| \geq 2 \tag{2.4}$$
$$e_j^2 = \beta e_j \tag{2.5}$$
$$e_j e_{j\pm1} e_j = e_j \tag{2.6}$$
$$e_i e_j = e_j e_i \qquad |i-j| \geq 2 \tag{2.7}$$
$$e_j b_{j\pm1} b_j = b_{j\pm1} b_j e_{j\pm1} = e_j e_{j\pm1} \tag{2.8}$$
$$b_j e_j = e_j b_j = \omega e_j \tag{2.9}$$

The functions f and g are polynomials. The parameter β is called the loop and the parameter ω the twist.

The braid generators $b_j^{\pm1}$ act on a set of N parallel strings by transposing the j and $j+1$ strands thereby imparting a half-twist. The braids can therefore be represented pictorially as follows



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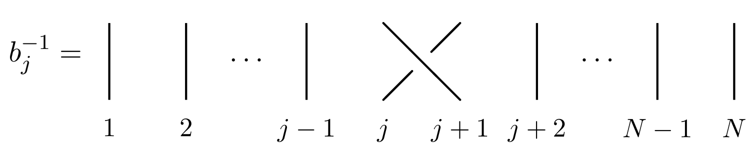

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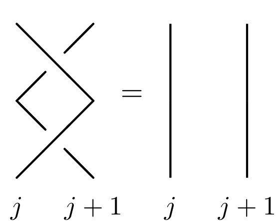
The defining relations then have a graphical interpretation where pictures related by continuous deformations of strings are considered equivalent:



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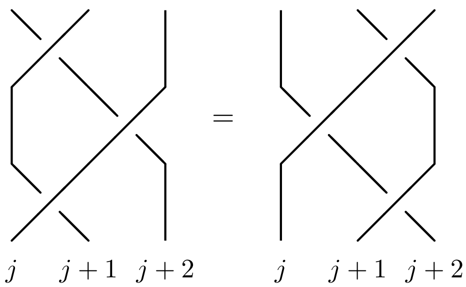

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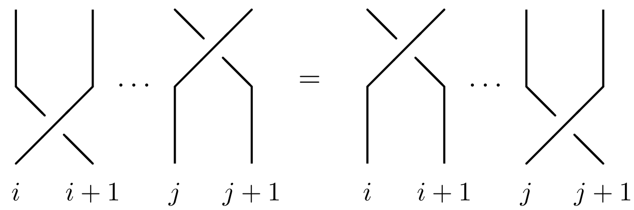

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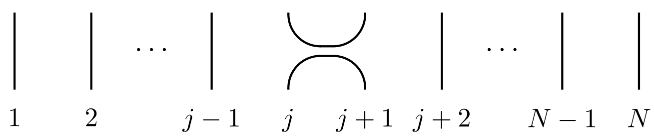
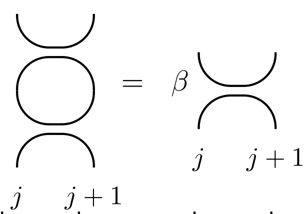
The Temperley-Lieb operators can be represented by a monoid diagram

$e_j =$ 

(2.15)

The defining relations therefore have the graphical interpretation



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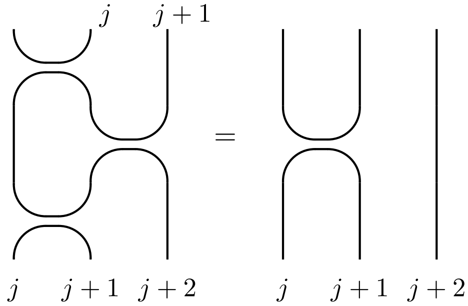

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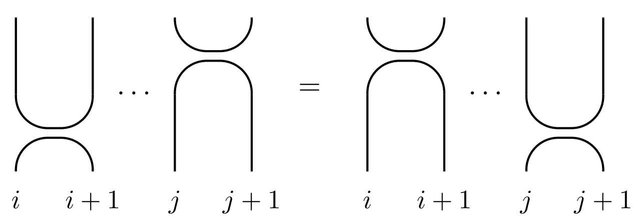
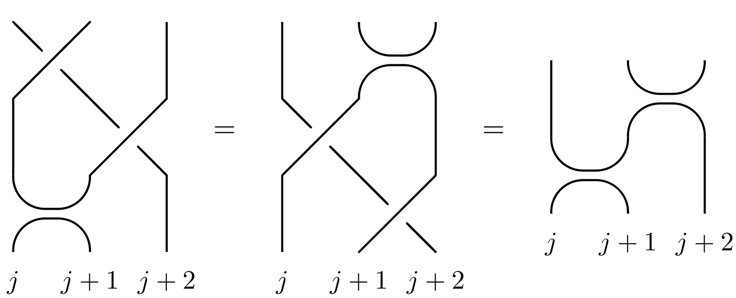


(2.18)

The relations involving both b_j and e_j also have graphical interpretations.



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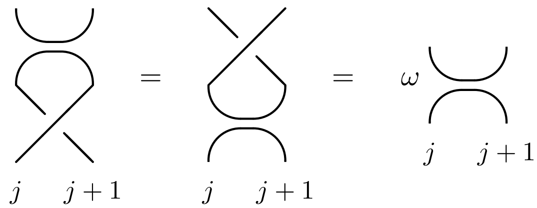
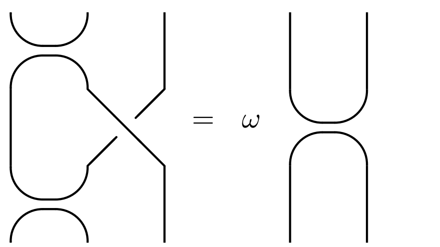


(2.20)

The following relations are immediate consequences of the defining relations of the braid-monoid algebra

$$
\begin{aligned}
b_{j\pm1}e_j &= b_j^{-1}e_{j\pm1}e_j && (2.21)\\
e_jb_{j\pm1} &= e_je_{j\pm1}b_j^{-1} && (2.22)\\
b_{j\pm1}e_je_{j\pm1} &= b_j^{-1}e_{j\pm1} && (2.23)\\
e_{j\pm1}e_jb_{j\pm1} &= e_{j\pm1}b_j^{-1} && (2.24)\\
b_{j\pm1}e_jb_{j\pm1} &= b_j^{-1}e_{j\pm1}b_j^{-1} && (2.25)\\
e_jb_{j\pm1}e_j &= \omega^{-1}e_j && (2.26)\\
e_jb_{j\pm1}^{-1}e_j &= \omega e_j && (2.27)
\end{aligned}
$$

These also admit graphical interpretations. For example the last equation becomes



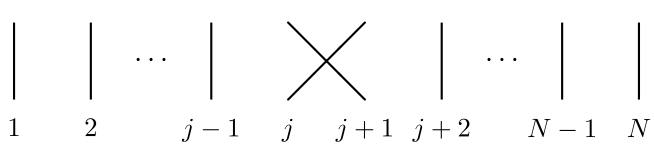
(2.28)

2.1 Brauer algebra

The Brauer algebra is generated by invertible braid operators b_j and Temperley-Lieb operators e_j with $j = 1, 2, \ldots, N-1$ satisfying

$$
\begin{aligned}
b_j^2 &= I && (2.29)\\
b_jb_{j+1}b_j &= b_{j+1}b_jb_{j+1} && (2.30)\\
b_ib_j &= b_jb_i, \qquad |i-j| \geq 2 && (2.31)\\
e_j^2 &= \beta e_j && (2.32)\\
e_je_{j\pm1}e_j &= e_j && (2.33)\\
e_ie_j &= e_je_i, \qquad |i-j| \geq 2 && (2.34)\\
b_je_j = e_jb_j &= e_j && (2.35)\\
b_ie_j &= e_jb_i, \qquad |i-j| \geq 2 && (2.36)\\
b_je_{j+1}e_j &= b_{j+1}e_j && (2.37)\\
e_{j+1}e_jb_{j+1} &= e_{j+1}b_j && (2.38)
\end{aligned}
$$

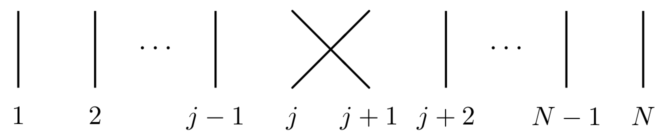
Diagrammatically, it is generated by a braid with no over-crossing.

$b_j =$ 

(2.39)

In accordance with relation (2.30), the inverse braid is identical as over- and under-crossings are identified.

$$b_j^{-1} = \qquad (2.40)$$



2.2 Brauer words

A word in the Brauer algebra is a sequence of generators of the form $g_{j_1} g_{j_2} \ldots g_{j_k}$ where $g_i \in \{b_i, e_i\}$ and $i \in \{1, \ldots, N-1\}$. These words can be reduced using the defining relations of the Brauer algebra. Thus for every word there exists an equivalent smallest word of the same or smaller length. For given N, these smallest words make up a basis , and the number of words in the basis is given by $(2N-1)!!$. Below are example bases for $N = 1, 2, 3$ and their corresponding dimensions.

$N = 1:$	$I,$	1
$N = 2:$	$I, e_1, b_1,$	3
$N = 3:$	$I, e_1, e_2, b_1, b_2, e_2e_1, e_1e_2, b_1e_2, e_2b_1, b_2b_1, b_1b_2, b_2e_1, e_1b_2, b_2e_1b_2, b_1b_2b_1,$	15

2.3 Birman-Wenzl-Murakami algebra

The Birman-Wenzl-Murakami (BMW) algebra is a braid-monoid algebra with f and g defined as

$$f(b_j) = (b_j + yI)(b_j - x^2yI)(b_j - y^{-1}I) = 0 \qquad (2.41)$$

$$e_j = g(b_j) = I + \frac{(b_j - b_j^{-1})}{(y - y^{-1})} \qquad (2.42)$$

where x and y are parameters, and the second equation is known as the skein relation. For convenience the skein relation is often written in the following form.

$$b_j = b_j^{-1} + \alpha(e_j - I) \qquad (2.43)$$

where $\alpha = y - y^{-1}$.
Multiplying both sides of the skein relation (2.42) by b_j gives

$$b_je_j = b_jI + \frac{b_jb_j - b_jb_j^{-1}}{\alpha} \qquad (2.44)$$

$$\omega e_j = b_j + \frac{b_j^2 - I}{\alpha} \qquad (2.45)$$

$$e_j = \frac{b_j^2 + \alpha b_j - I}{\alpha\omega} \qquad (2.46)$$

which shows that e_j can be written as a quadratic in b_j.
Using the cubic relation, it then follows by factorisation that

$$e_j^2 = (1 + \frac{x^2y - x^{-2}y^{-1}}{y - y^{-1}})e_j \qquad (2.47)$$

and

$$b_je_j = e_jb_j = x^2ye_j \qquad (2.48)$$

Therefore, $\{b_j, e_j | j = 1, 2, 3, \ldots, N-1\}$ generates a braid-monoid algebra with

$$\beta = 1 + \frac{x^2y - x^{-2}y^{-1}}{y - y^{-1}}, \qquad \omega = x^2y \qquad (2.49)$$

2.4 Birman-Wenzl-Murakami words

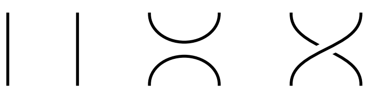
Similarly to the Brauer case, we can construct words with the generators b_j and e_j in the BMW algebra. Once again, every possible word is either a smallest word or can be reduced to one using the defining relations of the algebra. Thus we have a basis of words for the BMW algebra with dimension given by $(2N-1)!!$. In the diagrammatic representation of these words we can label each string by a number corresponding to the smallest node to which the string is connected. We call this number the *layer*. The nodes are labelled as $1, \dots, N$ left to right along the bottom and $N+1, \dots, 2N$ right to left along the top. We choose this basis to be in layers such that the string on layer 1 passes over all other strings, the string on the next highest layer passes over all strings except that on layer 1 etc. We can choose a basis in this way as every crossing can be replaced by the opposite crossing by applying the skein relation (2.43). Below are example bases for $N = 2, 3$ and their corresponding dimensions.

$N=1: \quad I, \qquad 1$
$N=2: \quad I, e_1, b_1, \qquad 3$
$N=3: \quad I, e_1, e_2, b_1, b_2, e_2e_1, e_1e_2, b_1e_2, e_2b_1, b_2b_1, b_1b_2, b_1e_2e_1, e_1e_2b_1, b_1e_2b_1, b_1b_2b_1, \qquad 15$

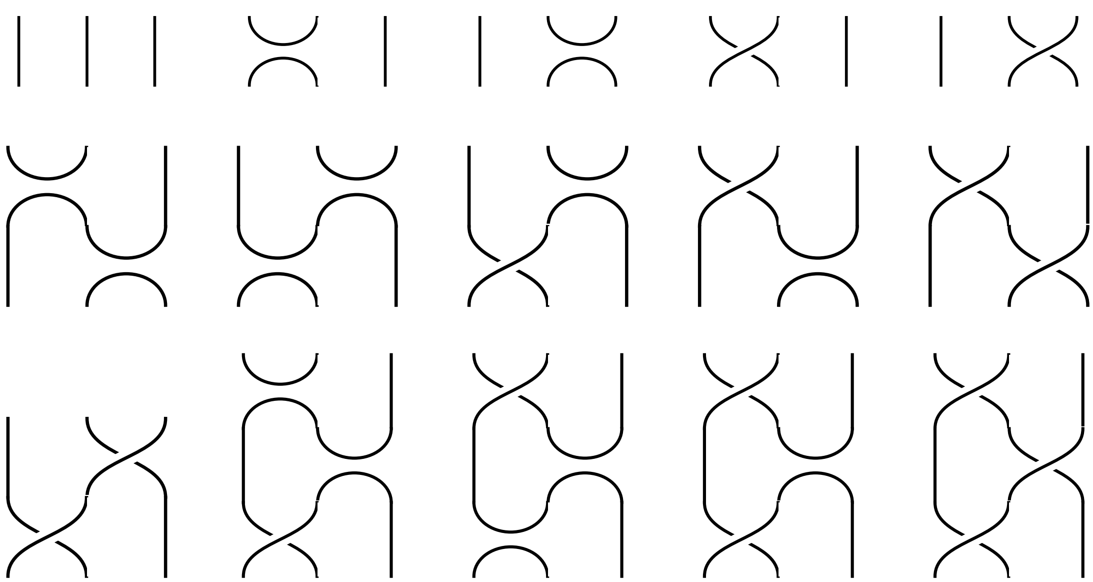
This can also be represented diagrammatically as follows.

$N=1$

$N=2$



$N=3$



The above bases can also be taken from the Brauer case by replacing every b_j in the Brauer algebra with a b_j or b_j^{-1} from the BMW algebra and ensuring that the resulting word is in layers. The BMW relations can then be used to remove the b_j^{-1} terms.

2.5 Brauer Limit

In the BMW algebra we take the limit $y^2 \to 1$ and $x^4 \to 1$. The skein relation (2.43) then gives

$$\begin{aligned}
\lim_{y\to\pm1} b_j &= \lim_{y\to\pm1}(b_j^{-1} + (y-y^{-1})(e_j - I)) && (2.50)\\
b_j &= b_j^{-1} + (\lim_{y\to\pm1}(y-y^{-1}))(e_j - I) && (2.51)\\
b_j &= b_j^{-1} && (2.52)
\end{aligned}$$

Taking these limits on equation (2.49) gives

$$\begin{aligned}
\lim_{(x^2,y)\to(\pm1,\pm1)} \beta &= \lim_{(x^2,y)\to(\pm1,\pm1)} (1 + \frac{x^2y - x^{-2}y^{-1}}{y-y^{-1}}) && (2.53)\\
&= 1 + \lim_{(x^2,y)\to(\pm1,\pm1)} \frac{x^2y - x^{-2}y^{-1}}{y-y^{-1}} && (2.54)\\
&= 1 + \frac{0}{0} && (2.55)
\end{aligned}$$

where this occurs for each of the four possible limits.
Therefore, β is indeterminate, which means it is a free parameter.

Similarily,

$$\lim_{(x^2,y)\to(\pm1,\pm1)} \omega = \lim_{(x^2,y)\to(\pm1,\pm1)} x^2y = \pm1 \tag{2.56}$$

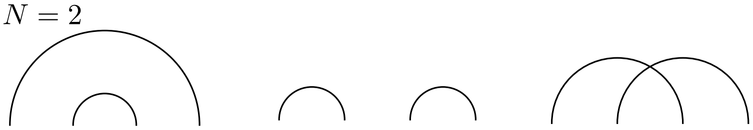
Taking $b_j = b_j^{-1}$ causes the BMW relations to collapse to the Brauer relations.

3 Link States

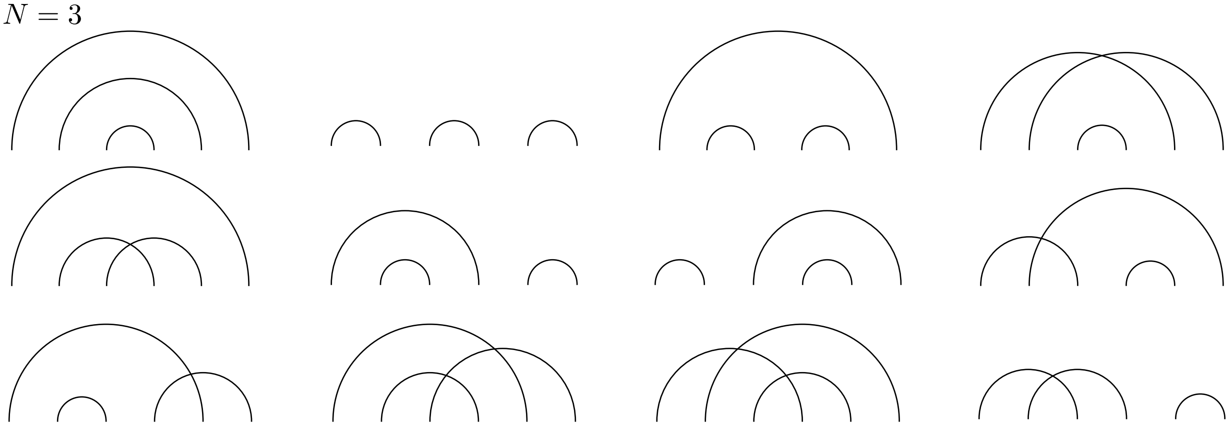
3.1 Link states as perfect matchings

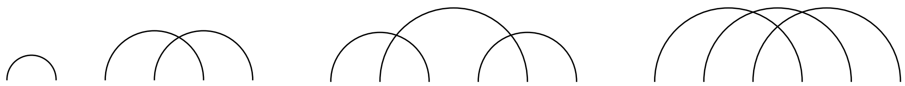
Link states are the perfect matchings of $2N$ nodes. Below are examples for $N = 2, 3$.

$N = 2$



$N = 3$

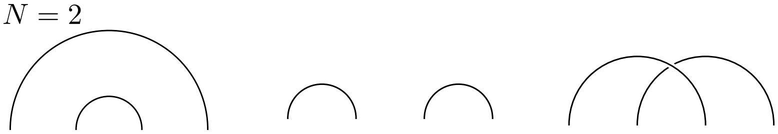




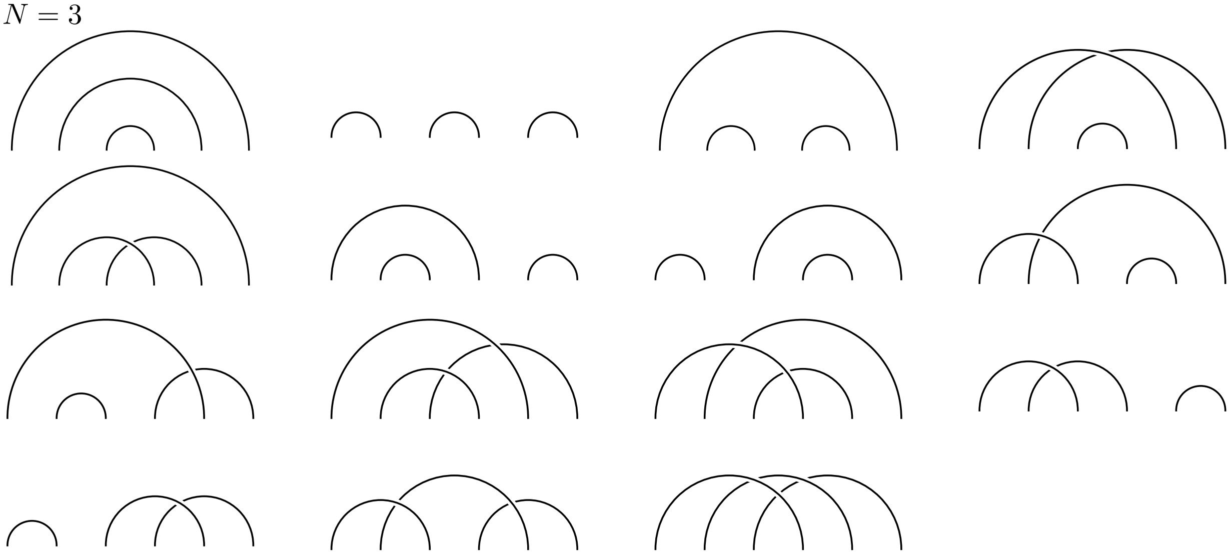
This is the Brauer case as there are no over- or under-crossings, only intersections. There are $(2N-1)!!$ link states.

By replacing each intersection in the Brauer case with an over-crossing, we obtain a basis for the link states in the BMW. Thus there are $(2N-1)!!$ basis link states. By an over-crossing we mean that each link must go over any link that starts to its right. This means that every basis link state can be separated into layers, where the layer of a link is the number of the smallest of the two nodes it connects. For example, for $N = 2, 3$ we have

$N = 2$



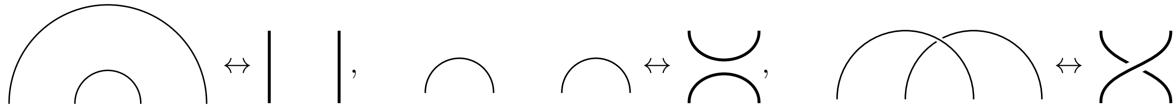
$N = 3$



Any crossing that causes a link state to not be separable into layers can be replaced locally using the skein relation (2.43). Doing this replacement repeatedly and simplifying will eventually result in a decomposition into the basis link states.

3.2 Bijection between words and link states

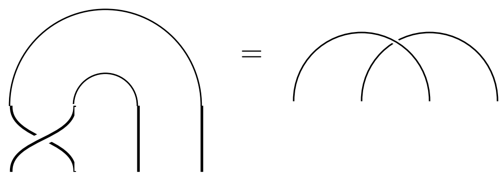
There is a bijection between the basis words and the basis link states in the BMW algebra. This is most easily seen diagrammatically. By taking a link state and stretching nodes $N+1$ to $2N$ up above nodes 1 to N we see that the diagrams for the link states are equivalent to those of the basis words up to continuous deformation. For example, for $N = 2$ we have,



4 Matrix Representations

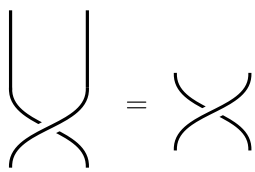
4.1 Algorithm for the action of the braids on link states

By acting with a braid on a basis link state in the BMW, we obtain another link state. This link state is either a basis link state or can be simplified to a linear combination of basis link states using the defining relations of the BMW. When acting on a basis link state on $2N$ nodes, we can act with b_j, e_j, where $j \in \{1, \ldots, 2N-1\}$. For example, acting with b_1 on the first basis link state for $N = 2$ gives the third basis link state, as shown below.



(4.1)

As discussed in section 3.2, there is a bijection between the BMW basis link states and basis words. When acting on basis words on N strings, we can act with b_j, e_j, where $j \in \{1, \ldots, 2N-1\}$. This means that the action of a braid on a link state can also be done in terms of basis words. Therefore, the previous equation is equivalent to



(4.2)

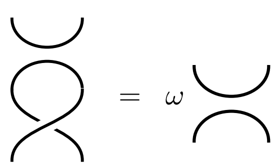
When acting with b_j where $j \in \{1 \ldots, N-1\}$, we can act with b_j to the left of the basis words. When acting with b_j where $j \in \{N+1, \ldots, 2N-1\}$ we act with b_{2N-j} to the right of the basis word. This is because we must rotate the braid and act with it on the to of the diagram, since nodes $N+1, \ldots, 2N-1$ have been brought around to the top. We still act with a braid as it is unchanged by a rotation of π. To act with b_N symbolically, we need to introduce a spectator term e_N to the right and then act to the left with b_N. After simplification, the spectator e_N can be removed to give a result on $2N$ nodes. Diagrammatically, we pull down node $N+1$ and introduce a monoid across nodes $N+2$ and $N+3$. We are now able to act with a b_N as we have $2N+2$ nodes instead of $2N$. After simplification, we pull node $N+1$ back to the top and remove the spectator monoid between nodes $N+2$ and $N+3$.

The following is an example in $N = 2$ acting with b_1, b_2 and b_3 on the basis word e_1 both symbolically and diagrammatically.

$N = 2$

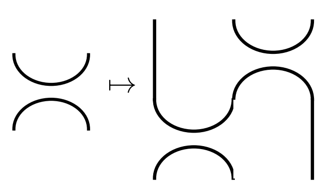
$j = 1$

$$b_1 e_1 = \omega e_1, \tag{4.3}$$

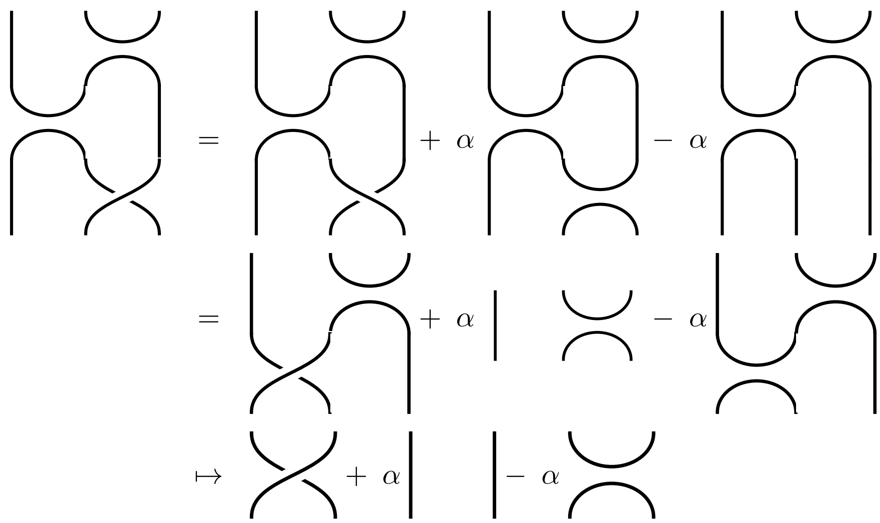


$j = 2 = N$

$$e_1 \mapsto e_1 e_2, \tag{4.4}$$



$$
\begin{aligned}
b_2 e_1 e_2 &= b_2^{-1} e_1 e_2 + \alpha\, b_2 e_1 e_2 - \alpha I e_1 e_2 = b_1 e_2 + \alpha e_2 - \alpha e_1 e_2 && (4.5)\\
&\mapsto b_1 + \alpha I - \alpha e_1 && (4.6)
\end{aligned}
$$



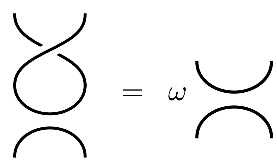
(4.7)

(4.8)

(4.9)

$j = 3$

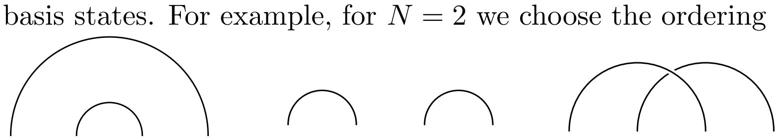
$$e_1 b_1 = \omega e_1,$$



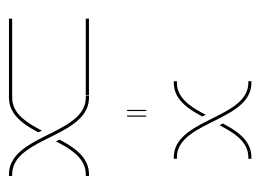
(4.10)

4.2 Construction of the matrix representatives

By acting on each of the basis link states with the generators b_j and e_j, we can find their matrix representations. The columns of the matrix represent the input basis states and the rows the output basis states. For example, for $N = 2$ we choose the ordering



or I, e_1, b_1 if we represent them by the corresponding basis words. To enter the first column of the matrix representation of b_1, we act b_1 on the first basis link state. Instead of acting on the link state itself, we can act on its corresponding basis word, I.

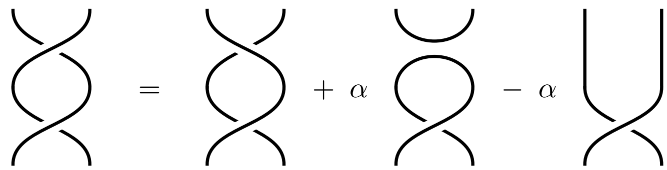


(4.11)

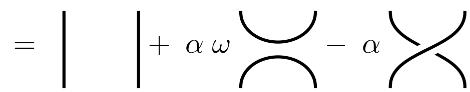
The output is the basis word b_1, so we fill the third row of the first column with a 1 to denote the coefficient of the output basis word. The rest of the first column contains zeros as the output did not contain the elements I or e_j.

$$
b_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left(\begin{array}{ccc} 0 & & \\ 0 & & \\ 1 & & \end{array}\right) \end{array}
$$

To fill the third column, we act with b_1 on the basis word b_1. This gives rise to a non-basis word b_1^2, requiring the use of the skein relation (2.43) on the second braid to decompose into basis words. We know that the second braid must be replaced as this is where the string on layer 2 passes over the string on layer 1, which is not allowed in our chosen basis.



(4.12)



(4.13)

The output is a linear combination of I, e_1 and b_1, so in column three the first row has a factor of 1, the second a factor of $\alpha\omega$ and the third a factor of $-\alpha$.

$$b_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega & \alpha\omega \\ 1 & 0 & -\alpha \end{pmatrix} \end{array}$$

For column two, the action of b_1 on e_1 gives the term ωe_1, so we place an ω in the second row. The diagrammatic working for this can be seen in equation (4.3).

4.3 Explicit 3×3 and 15×15 matrices for $N = 2, 3$

Below are the matrix representations for b_j and e_j for $N = 2, 3$, where ordering of the basis link states is the same as the ordering given in section 3.1.

$N = 2$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega & \alpha\omega \\ 1 & 0 & -\alpha \end{pmatrix}, \; b_2 = \begin{pmatrix} \omega & \alpha & 0 \\ 0 & -\alpha & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; b_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \omega & \alpha\omega \\ 1 & 0 & -\alpha \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \beta & \omega \\ 0 & 0 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} \beta & 1 & \omega^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \beta & \omega \\ 0 & 0 & 0 \end{pmatrix}$$

$N = 3$

$$b_1 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega & 0 & \alpha\omega & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha^2\omega & \alpha\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega & \alpha\omega & 0 & 0 & \alpha^2\omega & 0 & 0 & 0 & \alpha\omega \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\omega & 0 & \omega & \alpha\omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\alpha
\end{pmatrix}$$

$$b_2 = \left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \omega & 0 & \alpha\omega & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\omega \\
0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & \alpha\omega & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha
\end{array}\right)$$

$$b_3 = \left(\begin{array}{ccccccccccccccc}
\omega & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\
0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\omega & \alpha\omega & \alpha^2\omega & 0 \\
0 & 0 & -\alpha & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega & 0 & 0 & 0 & \alpha & \alpha & 0 & 0 & 0 & 0 & -\alpha^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)$$

$$b_4 = \left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \omega & 0 & \alpha\omega & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & \alpha\omega & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\alpha
\end{array}\right)$$

$$b_5 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega & 0 & \alpha\omega & 0 & 0 & 0 & 0 & 0 & -\alpha^2\omega & 0 & 0 & \alpha\omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & \alpha\omega & \alpha^2\omega & 0 & 0 & 0 & 0 & \alpha\omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\omega & 0 & \omega & 0 & \alpha\omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\alpha
\end{pmatrix}$$

$$e_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \beta & 0 & \omega & -\alpha & 1 & 0 & 0 & 0 & 0 & -\alpha\omega & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \alpha & 0 & \beta & \omega & 0 & 1 & \alpha\omega & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \omega & 0 & \beta & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$e_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \beta & 0 & \omega & 0 & 1 & \omega^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 1 & \omega^{-1} & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \beta & \omega & 0 & 0 & 1 & \omega^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$
e_3 = \begin{pmatrix}
\beta & 0 & 1 & 0 & \omega^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega^{-1} \\
0 & \beta & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \omega & \omega & \alpha\omega & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & 0 & 0 & 1 & 1 & \omega^{-1} & \omega^{-1} & 0 & 0 & -\alpha & -\alpha\omega^{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
e_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & \beta & 0 & \omega & 1 & 0 & 0 & \omega^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 1 & 0 & 0 & \omega^{-1} & 0 & \omega \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \beta & 0 & 0 & \omega & 1 & 0 & \omega^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
e_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \beta & 0 & \omega & -\alpha & 0 & 1 & 0 & 0 & -\alpha\omega & 0 & 0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \alpha & \beta & 0 & 0 & \omega & \alpha\omega & 1 & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \omega & 0 & \beta & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

5 Conclusion

The BMW algebra is an example of a braid-monoid or tangle algebra. It was introduced in the context of knot theory to study 2-d projections of knots through over-crossings and under-crossings. It is a non-commutative finite algebra without a natural distinguished basis and, therefore, no simple canonical form for its independent words. Therefore, it is non-trivial to do general algebra in the BMW.

Mathematically, a computer system has been implemented in Mathematica which can multiply arbitrary word in the algebra by the braid b_j. This means that the product of any two words in the algebra can be taken. The BMW generators were represented in Mathematica using a combination of the symbolic words and the connectivities obtained from the diagrammatic representations of the words. Layers were used to determine whether a word needed to be decomposed using the skein relation (2.43). When no further decomposition was required, the words were matched with the basis words using connectivities. In this way we were able to act with b_j on all basis words to obtain the loop representation of the BMW algebra.

Physically, this is related to exactly solvable 2-d lattice models. We can define face operators $X_j(u)$ in terms of the BMW generators.

$$X_j(u) = I + \eta^{-1}(z - z^{-1})(x^{-1}zb_j - xz^{-1}b_j^{-1}) \tag{5.1}$$

where

$$z = \exp(iu), \quad \eta = (x - x^{-1})(y - y^{-1}). \tag{5.2}$$

It can be shown that these face operators satisfy the Yang-Baxter equation

$$X_j(u)X_{j+1}(u + v)X_j(v) = X_{j+1}(v)X_j(u + v)X_{j+1}(u) \tag{5.3}$$

It is known that any representation of b_j in the BMW algebra leads to a solution of the Yang-Baxter equation. Remarkably, this famous equation expressing a local relation between face operators is sufficient to imply that it is related to a 2-d lattice model that is exactly solvable. It is a generalised model of polymers and percolation, since taking the Temperley-Lieb case with $\beta = 0$ gives a model for critical polymers and $\beta = 1$ gives a model for critical percolation.

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