Basic Real Analysis with Hyperreals without Transfer

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The existence of infinitesimals (infinitely small numbers) is an intuitive notion, so intuitive in fact that it has been around since the 3rd century BC. However, it wasn't until the 17th century that infinitesimals were most famously used in the development of calculus, in particular Leibniz's construction. Take for example his notation for the derivative of a function y(x): dy/dx. Very early on in our investigation of calculus as high school students, we are told that this is not a fraction, but that is exactly how Leibniz intended it to be interpreted – as a fraction. But then what are these differentials as he calls them, dy and dx? The answer: infinitesimals.

But what is an infinitesimal? By definition, a number is said to be infinitesimal if it is absolutely less than the reciprocal of every natural number. As the reals are an Archimedean field, it immediately follows that the only real infinitesimal is zero. However, above, dx cannot be zero, so how can it be infinitesimal? This is where my project begins.

In order to overcome the problem of existence of other infinitesimals, a new field is constructed, namely the hyperreals, which contains the real numbers as well as non-zero infinitesimals. Unlike the original use of the hyperreals by Abraham Robinson in 1960, I did not use mathematical logic, in particular, the Transfer Principle – a principle which allows a true statement of a certain logical form in the reals to be automatically true in the hyperreals, and vice versa. In order to avoid this principle, I had to consider the ultrapower construction of the hyperreals in which hyperreal numbers are equivalence classes of sequences of real numbers.

I finished by looking at a couple of basic concepts from real analysis such as continuity, convergence and differentiation. For each one, I was able to show that the (often intuitive) hyperreal interpretations of these concepts were equivalent to their standard definitions.

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See: www.ice-em.org.au/vs09.html