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Chromogeometry and conic sections

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I studied the behaviour of simple plane geometric objects—points, lines, circles, parabolas and general conic sections—when the traditional Euclidean distance measure $r^2 = x^2 + y^2$ is replaced by an arbitrary quadratic form. There are various motivations for such an approach. Some quadratic forms can be obtained by linear transformations of the Euclidean plane. Minkowski space (the geometry of special relativity) gives an example of a quadratic form which cannot be obtained by a linear transformation of Euclidean space. The theory works for an affine plane over any field, because no analytic notions are used.

I first investigated the isometries (quadratic form-preserving transformations) in these new geometries. For two special cases the isometries are easily parametrised: they are the rotations and reflections in Euclidean space, and the so-called “Lorentz boosts” in Minkowski space which result from changing one’s reference frame.

With conic sections defined according to the usual focus-directrix definitions, it turns out that essentially all results holding in Euclidean space have analogues over a general quadratic form, once perpendicularity and distance are suitably reformulated. After studying the basic algebraic properties of quadratic forms over a general field, I discovered that by extending the base field (for instance, from real to complex numbers), there is always an isometry from the new space back to Euclidean space. The one catch in this simple approach is that one must consider *null lines* (lines which are perpendicular to themselves), such as lines lying on the “light cone” in Minkowski space. When the field is extended, Euclidean space may gain some null lines (for instance, $x + iy + 1 = 0$) which usually have to be excluded from the statement of a theorem.

Conic sections always translate to second-degree polynomial equations in *x* and *y*. In general, the traditional subclasses of circles, ellipses, hyperbolas and so on are not preserved under a change of quadratic form, but there is one exception: the equation of a parabola under one quadratic form can only ever represent a parabola under any other quadratic form. Given a quadratic form, it turns out that there is a bijection between parabolas with non-null axes and possible choices of foci and directrices—an easy extension from the Euclidean case. The final section of the project consisted of an investigation of what happens to the parabola as the quadratic form itself is varied. The key observation is that the vertex is half way between the focus and the directrix along the axis.

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