Gamma-Convergence

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My AMSI project was on a topic in the calculus of variations called gamma-convergence. Essentially the work involved reading through lots of books and a few papers and trying to understand a number of proofs. First I had to learn about weak derivatives and Sobolev spaces and then about the direct methods of the calculus of variations and then some basic facts about gamma-convergence before I did some more advanced reading on some particular topics related to gamma-convergence, particularly homogenization and the localization method. I enjoyed being able to more-or-less direct my own studies and have decided to pursue an honours degree in financial mathematics this year.

In the calculus of variations we have a function of a function-typically an integral, which we call a functional, that we wish to minimize. For example, given two points A and B we may wish to find the shape of the curve that will cause a ball to roll from A to B under gravity in the least amount of time. In this case the functional is the amount of time and it's a function of the curve from A to B.

The problem of minimizing functionals was originally solved by determining certain differential equations, known as the Euler-Lagrange equations, that the minimizing solution (i.e. the curve of quickest descent in the above example) would satisfy reducing it to a problem in the well studied field of differential equations.

However eventually problems arose that couldn't be properly addressed with the Euler-Lagrange equations and their progeny leading to the introduction of a number of new techniques known as the direct methods of the calculus of variations. The basic idea is that if we can show our functional satisfies certain properties, especially what are called lower-semicontinuity and equi-coerciveness, then it must achieve its minimum- a fact that was notoriously difficult to establish with the older methods.

If we have a functional whose minimiser we are unable to determine it makes sense to consider a sequence of approximations to that functional and the minimisers of these approximations. The question is, in what manner must our sequence of functionals converge to our target functional in order for their minimisers to point wise converge to the minimiser of our target functional. The best answer is gamma-convergence, the associated mathematics of which then gives us a very powerful method of attacking minimum problems of functionals by making approximations. For a sequence of functionals $\{F_h\}$ we define the gamma upper and lower limits, respectively as follows:

$$\Gamma - \limsup(x) = \sup_{U \in N(x)} \limsup_{h \to \infty} \inf_{y \in U} F_h(y)$$

$$\Gamma - \liminf(x) = \sup_{U \in N(x)} \liminf_{h \to \infty} \inf_{y \in U} F_h(y)$$

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If these two functionals are equal, we denote them by F, we say that the sequence gamma-converges and we call F their gamma-limit.

Gamma-convergence forms the basis for the study of the Mumford-Shah functional which is used in computer science for image segmentation problems.

Gamma-convergence is also very important in materials science. It forms the basis of homogenization theory, which attempts to predict the properties of composite materials from the properties of their components, and is also fundamental in the study of phase transitions such as those that occur in the crystalline structure of shape-memory alloys. It also appears in studies of fluid flow through porous media (e.g. rocks) and in the design and mechanics of certain structures such as thin beams and shells.

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