# Projective Planes

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## 1. Introduction

The idea of using pairs of numbers to locate points in the plane and triples of numbers to locate points in space was first clearly expressed by Descartes in the seventeenth century. The concept was generalised to *Euclidean $n$-Space*, and eventually lead to the definition of abstract vector spaces over fields. The subject of vector spaces became so large and fundamental to modern mathematics that mathematicians tried to abstract away some of their properties to create ever more interesting structures. One such structure to emerge was the projective plane.

## 2. Projective Planes

Let $V$ be a three dimensional vector space over a field $F$ of order $q$. We call the one-dimensional subspaces of $V$ *points*, and the two-dimensional subspaces *lines*, with incidence given by the containment relation. This structure is denoted $PG(2, q)$.
A little linear algebra shows that the points and lines of $PG(2, q)$ satisfy:

1. every pair of distinct points are incident with a unique common line,
2. every pair of distinct lines are incident with a unique common point,
3. $PG(2, q)$ contains a set of four points such that no three are collinear.

We define a *Projective Plane* to be any set of abstract points and lines, together with an incidence relation that satisfies the above properties. In particular, $PG(2, q)$ is a *Desarguesian* projective plane.

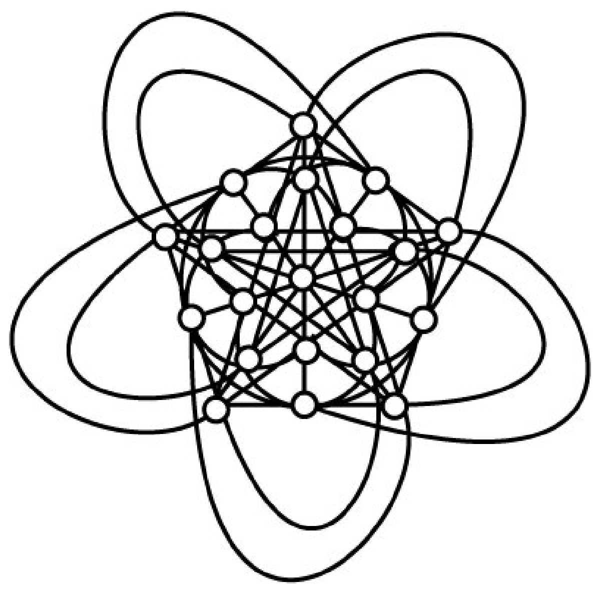

**Figure 1:** *A projective plane of order 4*

Given a projective plane $P$, is there a relationship between the number of points and lines of $P$? It turns out that as a simple consequence of the definition, there exists a number $n \leq 2$ (called the *order* of $P$) such that

1. each line contains exactly $n + 1$ points.
2. each point is on exactly $n + 1$ lines.
3. $P$ contains $n^2 + n + 1$ points and $n^2 + n + 1$ lines.

What is the order of $PG(2, q)$?

## 3. Ovals and Hyperovals

Let $P$ be a projective plane of finite order $n$. An *arc* of $P$ is a set of points, no three collinear. It is known that an arc of $P$ has size at most $n+2$, and equality occours only if $P$ has even order. In this case the arc is called a *hyperoval*. If $P$ has odd order, an arc of $P$ will have at most $n + 1$ points – such an arc is called an *oval*.
The classification of the ovals of $PG(2, q)$ is of fundamental importance in finite geometry. The remarkable theorem of Segre classifies all ovals of odd order; currently the problem is still open for $q \geq 64$, where $q = 2^h$. Try to find the ovals and hyperovals in the Fano Plane (Figure 2). How many are there? What about the projective plane shown in Figure 3.

## 4. Polarities

A *correlation* $\alpha$ of a projective plane is a one-to-one mapping of the points onto the lines and the lines onto the points such that $A$ is on $m$ if and only if $m^\alpha$ is on $A^\alpha$. A *polarity* is a correlation of order two (applying the polarity twice gives the identity map). Below are some examples of polarities, where like colours correspond to images.

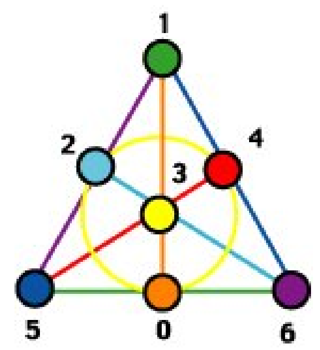


**Figure 2:** *A polarity of the* Fano Plane *(order 2)*

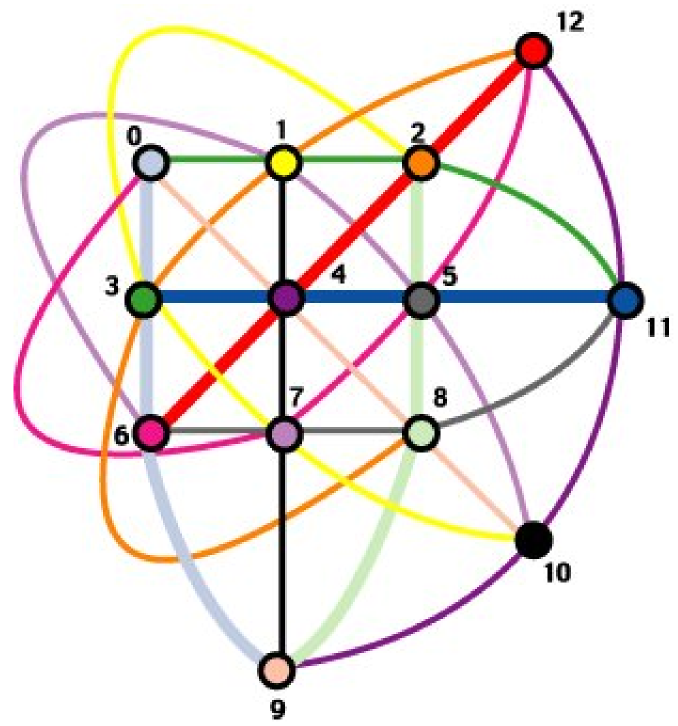


**Figure 3:** *A polarity of a projective plane of order 3*

When studying polarities we are often concerned with the points and lines that are incident with their image. These elements are the *absolute* points and lines of the polarity. The configuration of absolute points depends on the order of the plane on which they are defined. Let $\alpha$ be a polarity of a finite projective plane of non-square order $n$. Then $\alpha$ has $n + 1$ absolute points and

(a) if $n$ is even the absolute points are collinear,

(b) if $n$ is odd the absolute points form an oval.

Find the absolute points of the polarities shown above and convince yourself that they satisfy the required properties. What conditions do the absolute lines of a polarity satisfy?

## 5. An Application To Communication

Consider a set of telephone users who would like to speak with one another. In order for two users to speak to each other there needs to be a switch connecting them. Our goal is to minimize the number of switches.
Clearly we require at least one switch between any pair of users; since we are trying to minimize the number of switches we will further assume that any two users should be connected by exactly one switch. Additionally, we would like to be able use exactly the same hardware for each switch; this implies that each switch should connect the same number of users in the "same" way.
To solve this geometrically let users correspond to points and switches to lines. Our requirements translate to the following:

1. every pair of distinct points is incident with a unique line
2. every line is incident with the same number of points

It is clear that any projective plane satisfies the above constraints, so let us use the Fano Plane to construct such a communication system with 7 users.

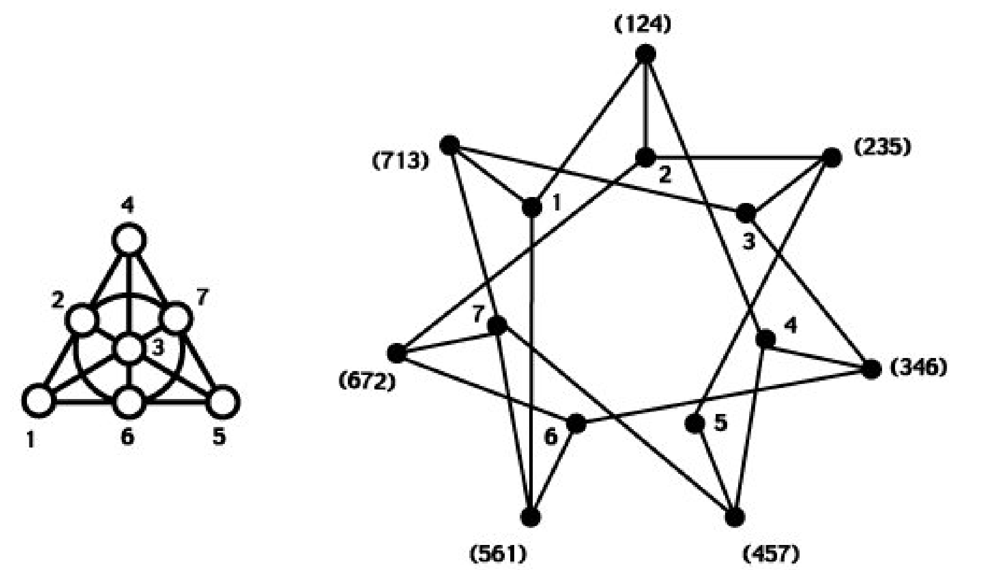


**Figure 4:** *The Fano plane with its incidence graph*

The figure on the right is the *incidence graph*, which in this application corresponds to our communication system. Each outer point represents a switch and each inner point represents a telephone user. Note that each switch connects users of the form $n,\ n+1 \pmod 7,\ n+3 \pmod 7$ (where $7 \equiv 0$). This is what was meant by saying that each switch must connect users in the same way.
So, given a system of seven users we can connect them using 7 switches, but is this the minimum? Try to construct a system that connects a different number of users in this way. What numbers are possible?