Quantum field theory and knot invariants

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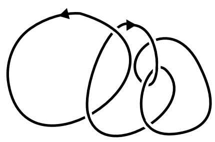
Outline

The aim of this project was to read Ed Witten's paper "Quantum field theory and the Jones polynomial" [9]. We also tried to motivate some of Witten's heuristics using 1D quantum mechanics. This necessitated reading in topology and geometry, gauge theory, classical mechanics, quantum mechanics, and quantum and conformal field theory.

The Jones polynomial

An *oriented link* is a set of disjoint, closed 1-manifolds embedded in $\mathbb{R}^3$, or its compatification $S^3 = \mathbb{R}^3 \cup \{\infty\}$. A link with two components is pictured below:



A *knot* is a link with one component. Two links, L_1 and L_2, are *isotopic* if there is some homeomorphism $f : S^3 \to S^3$, homotopic to the identity, mapping L_1 to L_2 and preserving the orientation of components.

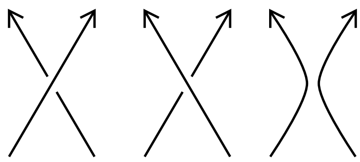
Let $\mathcal{L}$ be the set of oriented links. The *Jones polynomial* is an isotopy invariant $V : \mathcal{L} \to \mathbb{Z}[t^{\pm 1/2}]$, $L \mapsto V_L(t)$, defined by the condition

$$V_{\circlearrowright}(t) = 1,$$

where $\circlearrowright$ denotes the oriented unknot, and the *linear skein relations*

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t)$$

for any $L \in \mathcal{L}$. Here, L_+, L_-, and L_0 are three oriented links with diagrams identical to L except at one crossing, where they differ as below:



L_+ L_- L_0

To show that this is well defined, one uses induction on the number of crossings and invariance under Reidemeister moves. In a spectacular (but non-rigorous) tour de force, Witten recovers this invariant from quantum field theory.

Gauge theory

For interest, we give a brisk introduction to gauge theory. For a more leisurely development, we refer the reader to Nakahara [6]. Mathematical treatments can be found in Morgan [5] or Freed [2].

Let G be a Lie group. A *principal G-bundle* is a triple (P, B, π), P, B topological spaces and $\pi : P \to B$ a continuous surjection, with a right free action $\cdot : P \times G \to P$. We also require the existence of an open cover $\{U_\alpha\}$ of B such that for each α, there is a *trivialising map* $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ making the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times G \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ U_\alpha & & \end{array}$$

commute, where p_1 is projection onto the first factor. We usually write $P \overset{\pi}{\to} B$, $P \to B$, or $P(B, G)$ instead of the triple (P, B, π). A *section* on $P(U, G)$, $U \subseteq B$, is a map $s : U \to G$ such that $\pi \circ s = \mathrm{id}_U$.

Let $\mathfrak{g}$ be *the Lie algebra of* G, the vector space of left-invariant vector fields on G equipped with the Lie bracket

$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{g}.$$

We deal exclusively with finite-dimensional matrix Lie algebras. Let L_g and R_g denote left- and right-translation by $g \in G$, and set $\mathrm{Ad}_g = L_g \circ R_{g^{-1}}$. The *Maurer-Cartan form* $\theta : TG \to \mathfrak{g}$ is the canonical $\mathfrak{g}$-valued one-form which assigns to a vector $X \in TG$ the corresponding left-invariant vector field in $\mathfrak{g}$, i.e.,

$$L_g^* \theta X = \theta X, \quad \theta X|_e = X.$$

Given $P(B, G)$, $b \in B$, the fibre $\pi^{-1}(b) = P_b \simeq G$, so $T_p P_b \simeq \mathfrak{g}$ for any $p \in P_b$. Hence, we can pull back θ along the inclusion $i_b : P_b \to P$ to obtain $i_b^* \theta = \theta_b$.

We now introduce the main notions of gauge theory, *connections* and *curvature*. It was realised in the 1970s that these could be used to model (loosely speaking) the physical equivalence of different potentials and the associated fields.

Definition 1. *A* connection *on $P(B, G)$ is a $\mathfrak{g}$-valued one-form $A \in \Omega^1(P; \mathfrak{g})$ which satisfies*

1. $i_b^* A = \theta_b$, *for all* $b \in B$;
2. $R_g^* A = \mathrm{Ad}_{g^{-1}} A$, *for all* $g \in G$.

Denote the space of connections on P by $\mathcal{A}_P$. Given a local section σ and associated basis $\{dx^i\}$, we write $\sigma^* A = A_\mu \, dx^\mu$.

Definition 2. *The* curvature F_A *of $A \in \mathcal{A}_P$ is defined by*

$$F_A = dA + A \wedge A.$$

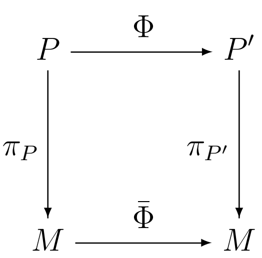
If $F_A = 0$, A is called a flat connection. *For a local section σ and associated basis $\{dx^i\}$,*

$$\sigma^* F = F_{\mu\nu} \, dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \, dx^\mu \wedge dx^\nu.$$

In the physics literature, $\sigma^* A$ and $\sigma^* F_A$ are called the gauge potential *and* field strength *respectively.*

Definition 3. *A G-*equivariant bundle isomorphism *between $P(B, G)$, $P'(B, G)$ is a diffeomorphism $\Phi : P \to P'$ such that*

1. $\Phi(p \cdot g) = \Phi(p) \cdot g$, *for all $g \in G$. Thus, Φ descends to $\bar{\Phi} : B \to B$.*
2. *Φ commutes with the projection,*



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We let $\mathcal{G}_P = \mathrm{Aut}(P)$ denote the set of G-equivariant bundle isomorphisms of P. This is often called the set of gauge transformations*.*

Theorem 4. *Let $P(B,G)$ be a G-bundle. Let $C^\infty(P,G)^G$ denote the set of G-equivariant maps from P to G, where G acts on itself by conjugation. Then $\mathcal{G}_P$ and $C^\infty(P,G)^G$ are isomorphic with $\Phi \mapsto g_\Phi$. Furthermore, if $g_\Phi = g$,*

$$\Phi^* A = \mathrm{Ad}_{g^{-1}} A + g^*\theta$$

where θ is the Maurer-Cartan form. Since G is a matrix group, $g^*\theta = g^{-1}\, dg$.

Proof. The correspondence $\mathcal{G}_P \to C^\infty(P,G)^G$ is given by

$$\Phi(p) = p \cdot g_\Phi(p).$$

The second equation is classical. See [6]. □

Chern-Simons theory

Witten exploits a heuristic relation between the Lagrangian and Hamiltonian formulation of a *topological quantum field theory* (TQFT) called *quantum Chern-Simons theory*. Let M be a 3-fold, G a compact, connected, simply-connected Lie group, $\mathfrak{g}$ the corresponding Lie algebra, and P a principal G-bundle over M. Define the *Chern-Simons action*

$$S_{CS}(A) = \frac{k}{4\pi}\int_M \mathrm{Tr}[A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A]$$

for a connection $A \in \Omega^1(P;\mathfrak{g})$ and a natural number $k \in \mathbb{Z}_{>0}$ called the *level* of the theory. The parameter k describes coupling strength and is used in the asymptotic study of Chern-Simons theory. This is integral under the action of G (i.e., defined up to an element of $\mathbb{R}/\mathbb{Z}$), so $e^{2\pi i k S(A)}$ is well defined. It is also a topological invariant. This basic setup is called *classical Chern-Simons theory*.

Let $\mathcal{A}_P/\mathcal{G}_P$ be the orbit space of gauge equivalence classes of connections over P, L a link with components $\{C_j\}$ tagged by representations $\{R_j\}$ of G, and $W_{R_j}(C_j)$ the R_j trace of the holonomy of C_j given a connection A. The latter are called *Wilson lines*. The Lagrangian version of quantum Chern-Simons theory is based on the heuristic *path integral*, aka *partition function*

$$Z(M;L) = \int_{\mathcal{A}_P/\mathcal{G}_P} DA\, e^{2\pi k i S_M(A)} \prod_{j=1}^{r} W_{R_j}(C_j).$$

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At present, this integral is not defined, since $\mathcal{A}_P/\mathcal{G}_P$ is a complicated, infinite-dimensional space; putting functional integrals on a rigorous footing is one of the key programs in contemporary mathematical physics.

In the Hamiltonian approach, we quantise the classical phase space, producing an associated *physical Hilbert space.* This is similar to canonical quantisation in elementary quantum mechanics, which we discuss at greater length below. Witten's particular strategy is called *geometric quantisation.* The classical phase space (the minima of the Chern-Simons action) may be shown to be the space of flat connections, i.e., connections A with

$$dA + A \wedge A = 0.$$

Though generally intractable, quantisation may be performed when $M = \Sigma \times \mathbb{R}$ for some Riemann surface Σ.

One can lift the natural symplectic form on the space of flat connections to the *symplectic quotient* $\mathcal{A}_P//\mathcal{G}_P$. This symplectic quotient is called the *moduli space of flat connections* $\mathcal{M}_\Sigma$ over Σ. It can also be shown that flat connections induce representations of the fundamental group of Σ in $\mathfrak{g}$, hence $\mathcal{M}_\Sigma$ is finite-dimensional.

Hilbert spaces and conformal field theory

In the absence of Wilson lines, the physical Hilbert space associated to $\mathcal{M}_\Sigma$ is the space of holomorphic sections of $L^{\otimes k}$, where L is the (projectively) flat determinant line bundle over $\mathcal{M}_\Sigma$. In the presence of Wilson lines, the physical Hilbert space is a more complicated object from *conformal field theory* (CFT) called the space of *conformal blocks.*

CFT is the study of fields (in the sense of quantum field theory) which are invariant under conformal diffeomorphisms. In 1+1 dimensions, we look at fields on the Riemann sphere which are invariant under the action of the Möbius group $\mathrm{PGL}(2, \mathbb{C})$. Using the *operator product expansion* (OPE) for two non-chiral, quasi-primary fields and the elementary theory of $\mathrm{PGL}(2, \mathbb{C})$, a 4-point operator $G(\mathbf{z}, \bar{\mathbf{z}})$, $\mathbf{z}, \bar{\mathbf{z}} \in \mathbb{C}^4$, may be expanded as a linear combination of functions related to the representations $\{R_j\}$. These functions are called conformal blocks and form a finite-dimensional vector space.

Recovering the Jones polynomial

Embed a link $L = \{C_j\}$ in S^3, with $G = \mathrm{SU}(n)$ and R_j the usual $\mathbb{C}^n$-representation of G. We draw a surface $\Sigma = S^2$ around a configuration corresponding to a crossing in a

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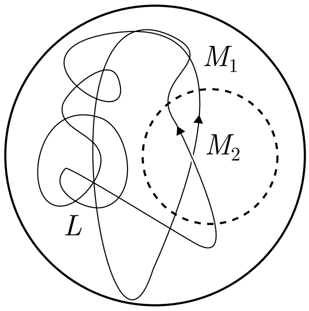
diagram of L. This splits S^3 into two parts, M_1 and M_2, which are both diffeomorphic to products $S^2 \times A$ for some $A \subseteq \mathbb{R}$.



The path integrals $Z(M_i; L|_{M_i})$, $i = 1, 2$, determine vectors ϕ, ψ in the Hilbert spaces $\mathcal{H}_i$ associated to the *boundaries* of the M_i. These have opposite orientations, so there is a natural pairing (ϕ, ψ), and from the general (heuristic) ideas of QFT

$$(\phi, \psi) = Z(S^3; L) \equiv Z(L).$$

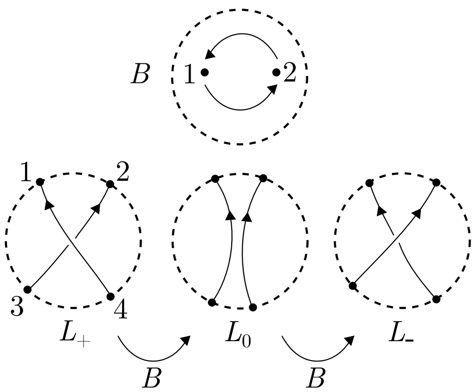
It can be shown that the $\mathcal{H}_i$ is 2-dimensional. If we rewire M_2 in two different ways (see below) we get two additional vectors ψ_1, ψ_2, so there is a relation

$$\alpha\psi + \beta\psi_1 + \gamma\psi_2 = 0.$$

Dotting with ϕ on the left, we get:

$$\alpha Z(L) + \beta Z(L_1) + \gamma Z(L_2) = 0.$$

Here, the L_i are obtained by gluing the rewired versions of M_2 back to M_1.



These three configurations differ from each other by a "twist" or "half monodromy" of the sphere called B (a diffeomorphism of S^2). With reference to the figure above, B swaps the strands at 1 and 2 by a ccw rotation; it fixes the strands at 3 and 4. B induces a linear operator on $\mathcal{H}_{S^2}$, which for simplicity we also call B. Since B operates on a 2-dimensional vector space, the Cayley-Hamilton theorem implies:

$$B^2 - B\,\mathrm{Tr}\,B + \det B = 0.$$

Further, we have $\psi_2 = B\psi_1 = B^2\psi$, so acting on ψ yields:

$$\det B \cdot \psi - \mathrm{Tr}\,B \cdot \psi_1 + \psi_2 = 0.$$

The operator B Dehn twists framings of links in S^3, so adding correction factors to recover the canonical framings gives:

$$\alpha = \det B, \quad \beta = e^{-\pi i(N^2-1)/N(N+k)}\,\mathrm{Tr}\,B, \quad \gamma = e^{-2\pi i(N^2-1)/N(N+k)}.$$

By Moore and Seiberg's results on B for $G = \mathrm{SU}(N)$, we have:

$$\alpha = -e^{2\pi i/(N(N+k))}, \quad \beta = -e^{i\pi(2-N-N^2)/N(N+k)} + e^{i\pi(2+N-N^2)/N(N+k)}.$$

We can divide out $e^{i\pi(N^2-2)/N(N+k)}$, and substitute $q = e^{2\pi i/(N+k)}$ to yield the relation:

$$-q^{N/2}Z(L) + (q^{1/2} - q^{-1/2})Z(L_1) + q^{-N/2}Z(L_2) = 0.$$

But in the terminology of the Jones polynomial, $L = L_+$, $L_1 = L_0$, and $L_2 = L_-$. Setting $N = 2$, we recover the linear skein relations, with $q = t$. It can also be shown that:

$$Z(\circlearrowright) = -\frac{\alpha + \beta}{\gamma} = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} = 1$$

for $N = 2$. Thus, the Jones polynomial may be viewed as the partition function of a quantum Chern-Simons theory with gauge group SU(2).

Research

One focus of my research was pedagogical—trying to understand Chern-Simons theory (and more generally TQFTs) by analogy with 1D quantum mechanics, and in particular, the connection between Hamiltonian and Lagrangian approaches.

We begin with the Hamiltonian. In the 1D case, we usually quantise a classical theory—a Hamiltonian formalism in terms of phase space (with a *configuration space* of position variables) and Poisson bracket $\{\cdot,\cdot\}$. To get the corresponding quantum theory, we *canonically quantise* by making the following replacements:

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- configuration space $\to$ Hilbert space $\mathcal{H}$ of functions on the configuration space
- variables $\to$ linear operators on $\mathcal{H}$
- $\{\cdot,\cdot\} \to -\frac{1}{i\hbar}[\cdot,\cdot]$.

See Shankar [8] for further details. This canonical quantisation is analogous to geometric quantisation in Chern-Simons theory. In Chern-Simons theory, however, the configuration space is replaced by the space of classical solutions (the flat connections) for a principal G-bundle over Σ, and the function space with a sequence of "lifts":

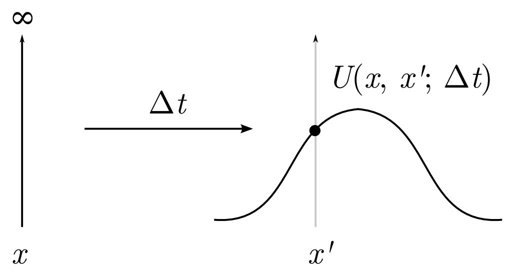
flat connections $\to$ moduli space $\mathcal{M}$ of flat connections $\to$ vector bundle over $\mathcal{M}$.

This is precisely the sequence of lifts needed to eliminate dependence on the complex structure of Σ.

We now consider the 1D version of the partition function. To find the dynamics in a 1D quantum system, we solve the *Schrödinger equation*

$$(i\hbar\partial_t - H)\psi = 0$$

where H is the quantum Hamiltonian operator. Usually, we first solve the *time independent Schrödinger equation*, yielding a Green's function called the *propagator* U. In the 1D case, the propagator $U(x', x; \Delta t)$ evolves a delta function at $x \in \mathbb{R}$ a time Δt using the Schrödinger equation, then samples the resulting wave function at x':



We can express an initial wave function as an integral of deltas and evolve them independently.

In the Lagrangian approach to 1D quantum mechanics, the propagator $U(x', x; \Delta t)$ is given directly as a path integral. Let $\mathcal{P}$ denote the set of continuous time-parameterised paths $\gamma : [0, \Delta t] \to \mathbb{R}$ with $\gamma(0) = x$ and $\gamma(\Delta t) = x'$. Suppose that we have an action $S : \mathcal{P} \to \mathbb{R}$ from the classical description. Then Schrödinger's equation implies that:

$$U(x', x; \Delta t) = \int_{\gamma\in\mathcal{P}} e^{S[i\gamma]}\, d\mathcal{P}.$$

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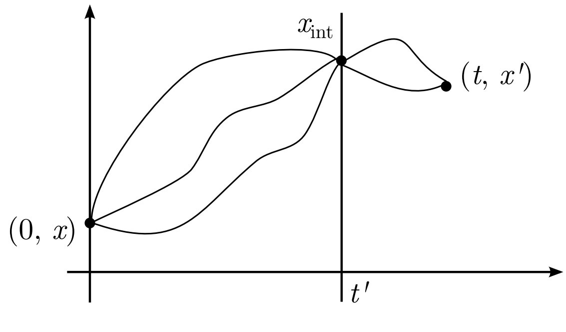
For a derivation, see [8]. The expression $e^{iS[\gamma]}$ is called the *phase* of the path. Of course, the formal structure of Witten's partition function is similar to this 1D path integral, with connections replacing paths, and the measure keeping track of the topology via Wilson lines.

Note that we can look at an intermediate time t' between 0 and t:



We can split a path between $(0, x)$ and $(\Delta t, x')$ at the vertical line at t'. Since the propagator is multiplicative, we factorise and then integrate over the point of intersection x_{int} to get

$$U(x', x; t) = \int_{\mathbb{R}} U(x_{\text{int}}, x; t')U(x', x_{\text{int}}; t - t')\, dx_{\text{int}}.$$

There are three notable features:

- we are using propagators on subspaces with shared boundary;
- we combine them with an *inner product*-like thing to get the full propagator;
- in the integrand, the functions $U(x_{\text{int}}, x; t')$ and $U(x', x_{\text{int}}; t - t')$ depend only on position x_{int}, hence live in our Hilbert space.

These three features also appear in Witten's theory. We combine partition functions on M_1 and M_2 with an inner product to recover the whole partition function; this inner product is defined because M_1 and M_2 share a boundary. Moreover, partition functions evaluate to elements of the corresponding physical Hilbert space. Thus, Witten's heuristic expectations for quantum Chern-Simons theory are not so different from elementary quantum mechanics!

Thinking of the diffeomorphism B as a *braid action* on the two strands, we considered the possibility that braid actions on n strands might have well-understood conformal representations. Using Witten's strategy, these could yield interesting skein relations between local "perturbations" of a link with n inputs. Unfortunately, we did not have time to develop these ideas further.

Acknowledgements

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I would also like to thank CSIRO and AMSI for organising the Big Day In 2013. This was a great opportunity to meet other young scientists and hear about cutting edge projects in mathematics, statistics, computer science, astronomy, and other areas. The heady mix of networking, technical content, and free coffee is not to be missed!

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