AMSI (2017)

**Stochastic Equations and Processes in Physics and Biology**

# Exercise sheet 3: Random walk and diffusion equation, the Wiener-Khinchin theorem, correlation function and power spectral density

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- 1. **Q1: Inhomogeneous biased random walk.** A walker steps a distance $\Delta$ with probabilities $p(x)$ and $q(x)$ to the right and to the left, respectively every $\delta t$ seconds. Show that the Fokker-Planck equation is given by

$$\partial_t P(x,t) = -\partial_x (f(x)P(x,t)) + D\partial_x^2 P(x,t),$$

with the drift force $f(x) = \Delta \frac{p(x)-q(x)}{\delta t}$ and the diffusion coefficient $D = \frac{\Delta^2}{2\delta t}$.

- 2. **Q2: Diffusion equation.** A one-dimensional domain is bounded by a wall at $x = 0$ and has a sink at $x = a$. You are releasing random walkers at $0 < x = x_0 < a$ with the rate five walkers every second. Determine the average number of walkers in the domain in the stationary state. Solve the problem in the continuous limit, assuming that the random walkers are symmetric and the diffusion coefficient is $D$.

- 3. **Q3: The Wiener-Khinchin theorem.** The power spectral density $S(\omega)$ of the process $s(t)$ is known. Find the corresponding stationary ACF for

(a)
$$S(\omega) = 1$$

(b)
$$S(\omega) = \begin{cases} 1 & \omega \le a, \\ 0 & \text{otherwise} \end{cases}$$

(c)
$$S(\omega) = \begin{cases} 1-\omega & \omega \le 1, \\ 0 & \text{otherwise} \end{cases}$$

- 4. **Q4: Power spectral density of a linear system.** Compute the power spectral density $S(\omega)$ for the following stochastic processes

(a)
$$\dot{x} = -\alpha x + \beta x(t-\tau) + D\xi(t)$$

(b) The Van der Pol oscillator:

$$\dot{x} = y, \;\; \dot{y} = -x - \alpha y + \beta y(t-\tau) + D\xi(t),$$

where $\xi(t)$ is the Gaussian white noise with $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$.

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**Q1: Solution**

The master equation is given by

$$P(x,t+\delta t) = p(x-\Delta)P(x-\Delta,t) + q(x+\Delta)P(x+\Delta,t)$$

Using Taylor series expansion

$$\begin{aligned}
P(x,t+\delta t) &\approx p(x,t) + \partial_t p(x,t)\delta t + \ldots, \\
p(x-\Delta)P(x-\Delta,t) &\approx p(x)P(x,t) - \partial_x(p(x)P(x,t))\Delta + \frac{\Delta^2}{2}\partial_x^2(p(x)P(x,t)) + \ldots, \\
q(x+\Delta)P(x+\Delta,t) &\approx q(x)P(x,t) + \partial_x(q(x)P(x,t))\Delta + \frac{\Delta^2}{2}\partial_x^2(q(x)P(x,t)) + \ldots
\end{aligned}$$

We obtain

$$\begin{aligned}
\partial_t P(x,t) &\approx \frac{1}{\delta t}\left[p(x)P(x,t) - \partial_x(p(x)P(x,t))\Delta + \frac{\Delta^2}{2}\partial_x^2(p(x)P(x,t))\right. \\
&+ \left. q(x)P(x,t) + \partial_x(q(x)P(x,t))\Delta + \frac{\Delta^2}{2}\partial_x^2(q(x)P(x,t)) - P(x,t)\right] \\
&= \partial_x\left[-\frac{p(x)-q(x)}{\delta t}P(x,t) + \frac{\Delta^2}{2\delta t}\partial_x P(x,t)\right].
\end{aligned}$$

## Q2: Solution

The entire domain $[0, a]$ is divided into two parts: $D_1 :\ [0, x_0]$ and $D_2 :\ [x_0, a]$. In the stationary regime, the walkers that move to the left from $x_0$ into the domain $D_1$ will eventually return to $x_0$, after bouncing off the wall at $x = 0$. This implies that the probability current $J_1 = -D\partial_x P_1(x)$ is zero in $D_1$. Therefore, the stationary density $P_1(x)$ in $D_1$ is constant

$$P_1(x) = C_1.$$

In the domain $D_2$ the current $J$ in the stationary regime is constant

$$J = -D\partial_x P_2(x).$$

Consequently,

$$P_2(x) = -\frac{J}{D}x + C_2,$$

where the constants $C_2$ can be determined from the boundary conditions. Namely, we require that the density is continuous at $x = x_0$ and that $P_2(x = a) = 0$ (absorbing boundary). This yields

$$C_1 = -\frac{J}{D}x_0 + C_2, \quad -\frac{J}{D}a + C_2 = 0.$$

Solving for $C_1$ and $C_2$, we obtan

$$P_1 = \frac{J}{D}(a - x_0), \quad P_2 = \frac{J}{D}(a - x).$$

The total number of walkers $N$ in the domain is found as

$$N = \int_0^{x_0} P_1(x)\, dx + \int_{x_0}^{a} P_2(x)\, dx = \frac{J}{D}\left[(a - x_0)x_0 + a(a - x_0) - \frac{a^2}{2} + \frac{x_0^2}{2}\right] = \frac{J}{2D}\left[a^2 - x_0^2\right].$$

The current $J$ gives the number of walkers that pass through the system per unit of time (per one second)

$$J = 5\left(\frac{\text{walkers}}{\text{second}}\right).$$

Under this condition, the number of walkers in the domain is

$$N = \frac{5}{2D}(a^2 - x_0^2).$$

**Q3: Solution**

(a)

$$G(\tau) = \int_{-\infty}^{\infty} e^{\pm i\omega\tau}\, d\omega = (2\pi)\delta(\tau)$$

(b)

$$\begin{aligned} G(\tau) &= 2\mathrm{Re}\left(\int_0^a e^{\pm i\omega\tau}\, d\omega\right) = 2\mathrm{Re}\left(\left.\frac{e^{i\omega\tau}}{i\tau}\right|_0^a\right) \\ &= 2\mathrm{Re}\left(\frac{e^{ia\tau}-1}{i\tau}\right) = \frac{2\sin(a\tau)}{\tau}. \end{aligned}$$

Note that Dirac's delta function can be represented as

$$\lim_{a\to\infty} \frac{\sin(ax)}{\pi x} = \delta(x).$$

The ACF is shown in Fig. 1 for three different values of $a$

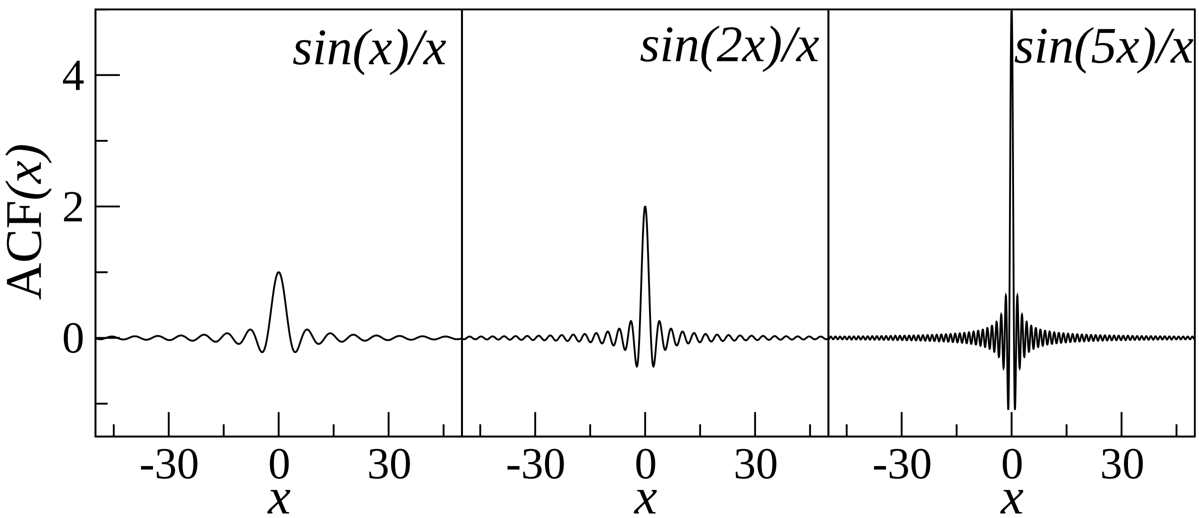


Figure 1: ACF from part (b) $\frac{2\sin(a\tau)}{\tau}$ for $a = 1, 2, 5$.

(c)

$$G(\tau) \quad = \quad 2\mathrm{Re}\left(\int_0^1 e^{\pm i\omega\tau}(1-\omega)\,d\omega\right) = 2\mathrm{Re}\left(\frac{1-e^{i\tau}+\tau i}{\tau^2}\right) = 2\frac{1-\cos\tau}{\tau^2}.$$

The ACF is shown in Fig. 2

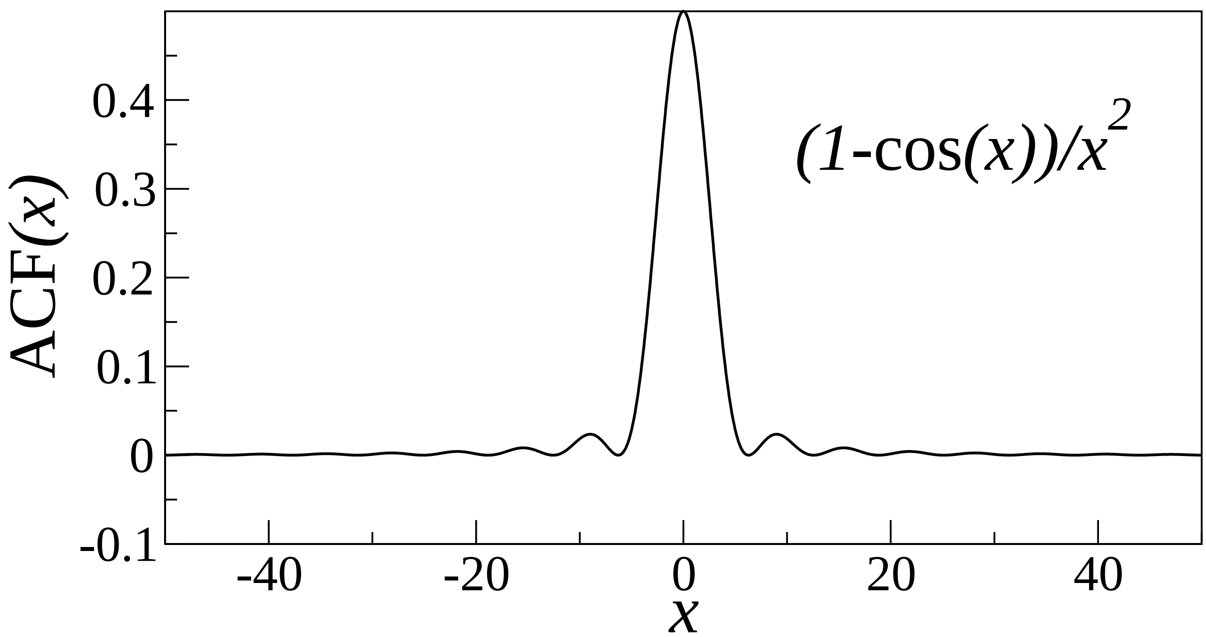


Figure 2: ACF from part (c) $\frac{1-\cos\tau}{\tau^2}$.

**Q4: Solution**

(a) In the Fourier space

$$i\omega\hat{x} = -\alpha\hat{x} + \beta e^{-i\omega\tau}\hat{x} + D\hat{\xi}$$

Solving for $\hat{x}$

$$\hat{x} = \frac{D\hat{\xi}}{i\omega + \alpha - \beta e^{-i\omega\tau}}.$$

Taking the modulus

$$S_x(\omega) = \frac{1}{2\pi}\frac{D^2}{(\alpha - \beta\cos(\omega\tau))^2 + (\omega + \beta\sin(\omega\tau))^2}$$

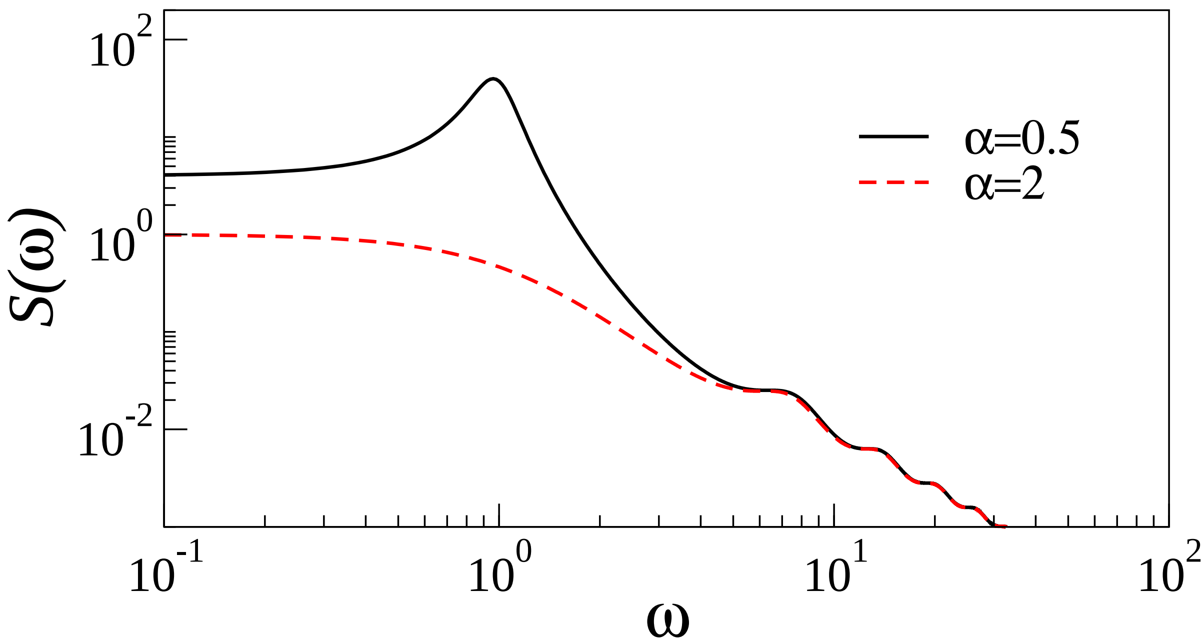


Figure 3: Power spectrum $S(\omega)$ from part (a) for $\beta = 1$, $\tau = 1$ and two different values of $\alpha$ as in the legend.

(b) **Van der Pol oscillator**
In the Fourier space

$$i\omega\hat{x} = \hat{y}, \;\; i\omega\hat{y} = -\hat{x} - \alpha\hat{y} + e^{-i\omega\tau}\beta\hat{y} + D\hat{\xi}.$$

Solving for $\hat{x}$ and $\hat{y}$

$$\hat{x} = \frac{D\hat{\xi}}{1 - \omega^2 + i\omega\alpha - i\beta\omega e^{(-i\omega\tau)}}$$
$$\hat{y} = i\omega\hat{x}.$$

The power spectrum of the $x$ coordinate

$$S_x(\omega) = \frac{1}{2\pi}\frac{D^2}{(1 - \omega^2 - \beta\omega\sin\omega\tau)^2 + \omega^2(\alpha - \beta\cos\omega\tau)^2}.$$

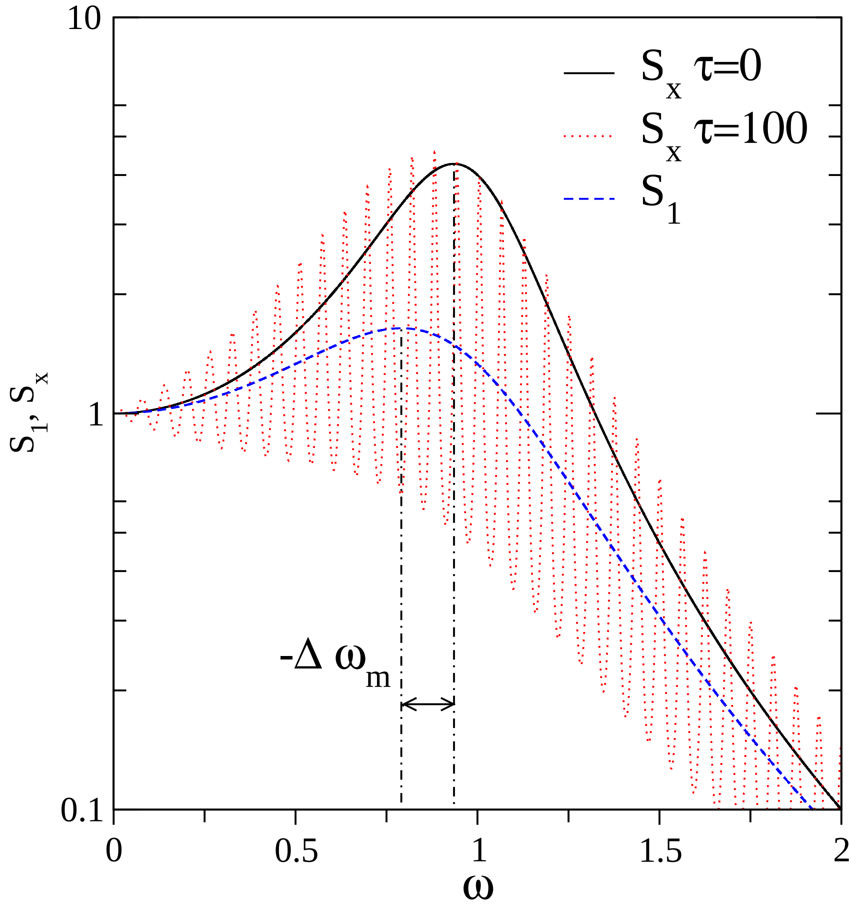


Figure 4: Power spectrum $S_x(\omega)$ for $\tau = 0$ (solid line), $\tau = 100$ (dotted line) and the background spectrum $S_1(\omega)$ (dashed line). Other parameters are $\beta = 0.5$ $\alpha = -1$.

The background function $S_1(\omega)$ can be determined as the limit of the running average of $S_x(\omega)$ over $2\pi/\tau$

$$S_1(\omega) = \lim_{\tau\to\infty}\frac{\tau}{2\pi}\int_{\omega}^{\omega+2\pi/\tau} S_x(\omega')\,d\omega'.$$

This yields

$$S_1(\omega) = \frac{D^2}{[\omega^2 - 1]^2 + \omega^2[(\alpha+\beta)^2 + 2(\alpha+\beta)\beta]}.$$

**ACF:**


- A. Pototsky and N. Janson, *Correlation theory of delayed feedback in stochastic systems below Andronov-Hopf bifurcation*, Phys. Rev. E **76**, 056208 (2007)
- Valentin Flunkert and Eckehard Schöll *Suppressing noise-induced intensity pulsations in semiconductor lasers by means of time-delayed feedback*, Phys. Rev. E **76**, 066202 (2007)


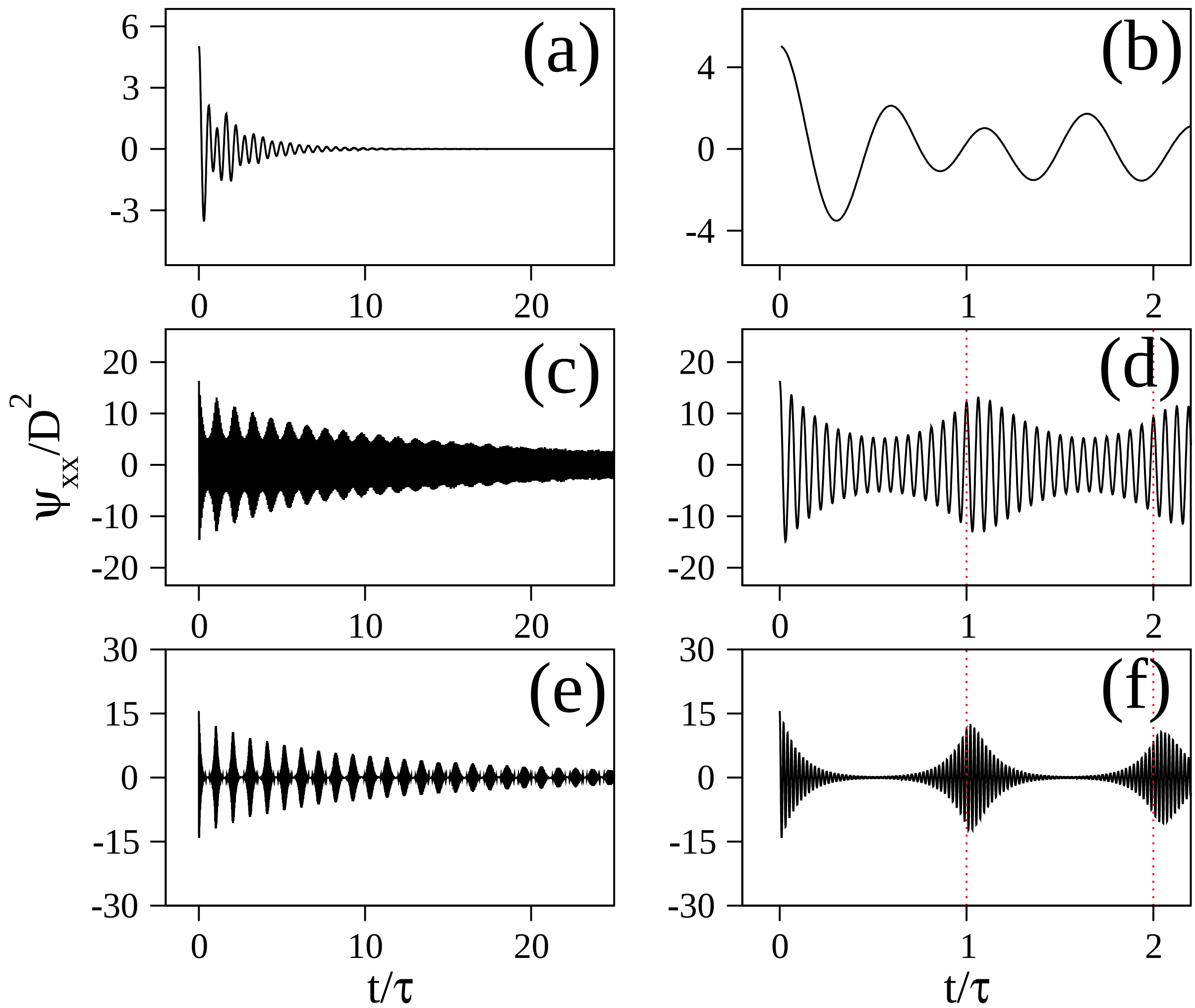


Figure 5: ACF of $x$ in the units of $D^2$ at different values of the delay time $\tau$: (a) and (b) $\tau = 10$, (c) and (d) $\tau = 100$, (e) and (f) $\tau = 300$. Panels (a), (c) and (e) show the behaviour of the ACF on the scale of $t = 25\tau$. Panels (b), (d) and (f) reveal the behaviour on the scale of $t = 2\tau$.