AMSI (2017)

Stochastic Equations and Processes in Physics and Biology

Exercise sheet 2: The random telegraph process, elements of the renewal theory

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- 1. **Q1: Stationary average $\langle s \rangle$ of a two-state renewal process**

  Stochastic process $s(t)$ is an alternating renewal process, where $s(t)$ is switching between two distinct values $s = a$ and $s = b$. The distribution of the residence times in the states $s = a$ and $s = b$ are given by $f_a(t)$ and $f_b(t)$, respectively. The time-average value of $s(t)$ is calculated according to $\langle s \rangle = \lim_{T\to\infty}(1/T)\int_0^T s(t)\,dt$

  (a) Show that
  $$\langle s \rangle = \frac{a\langle t_a \rangle + b\langle t_b \rangle}{\langle t_a \rangle + \langle t_b \rangle},$$
  where $\langle t_{a,b} \rangle = \int_0^\infty t f_{a,b}(t)\,dt$.

  (b) Further show that for the exponential distribution of the residence times
  $$f_a(t) = \lambda e^{-\lambda t}, \;\; f_b(t) = \mu e^{-\mu t}$$
  The time-average $\langle s \rangle = \lim_{T\to\infty}(1/T)\int_0^T s(t)\,dt$ coincides with the stationary ensemble average
  $$\langle s \rangle = \frac{a\mu + b\lambda}{\lambda + \mu}.$$

- 2. **Q2: time average *vs* ensemble average ACF.** For a random telegraph process $s(t) = \pm 1$ with the transition rates $\lambda = \mu$ show that the ACF computed from a single realization
  $$G(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T s(t)s(t+\tau)\,dt$$
  coincides with the stationary ensemble-averaged ACF
  $$G(\tau) = e^{-2\lambda\tau}.$$

- 3. **Q3: Waiting for bus problem.** You arrive at a bus stop at a randomly chosen moment of time. What is the average waiting time for the next bus? Assume that the buses arrive at the bus stop at random intervals of time. The distribution density function (pdf) of the time interval t between two subsequent arrivals is known and given by $f(t)$.

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Q1: Solution

Assume that over the interval T there were $n+1$ switches between $s=a$ and $s=b$. Without loss of generality we can assume that at $t=0$ the system was in the state $s=a$. Then the time-average of $s(t)$ is given by

$$\langle s\rangle = \frac{a(t_a^{(1)}+t_a^{(2)}+\ldots+t_a^{(n)})+b(t_b^{(1)}+t_b^{(2)}+\ldots+t_b^{(n)})}{(t_a^{(1)}+t_a^{(2)}+\ldots+t_a^{(n)})+(t_b^{(1)}+t_b^{(2)}+\ldots+t_b^{(n)})},$$

where $t_a^{(i)}$ denotes the interval of time spent in the states a before the i-th switch and $t_b^{(i)}$ denotes the interval of time spent in the states b after i-th switch.

Consequently,

$$\langle s\rangle = \frac{a(t_1^{(a)}+t_2^{(a)}+\ldots+t_n^{(a)})/n+b(t_1^{(b)}+t_2^{(n)}+\ldots+t_n^{(b)})/n}{(t_1^{(a)}+t_2^{(a)}+\ldots+t_n^{(a)})/n+(t_1^{(b)}+t_2^{(b)}+\ldots+t_n^{(b)})/n} = \frac{a\langle t_a\rangle+b\langle t_b\rangle}{\langle t_a\rangle+\langle t_b\rangle},$$

where $\langle t_{a,b}\rangle = (1/n)\sum_{i=1}^n t_{a,b}^{(i)}$. In the limit of large number of switches, the arithmetic average coincides with the expected value

$$\lim_{n\to\infty}(1/n)\sum_{i=1}^n t_{a,b}^{(i)} = \int_0^\infty t f_{a,b}(t)\,dt.$$

For the random telegraph process

$$\langle t_a\rangle = \int_0^\infty dt\, t\lambda e^{-\lambda t} = \frac{1}{\lambda}$$

$$\langle t_b\rangle = \int_0^\infty dt\, t\mu e^{-\mu t} = \frac{1}{\mu}$$

Finally

$$\langle s\rangle = \frac{a/\lambda+b/\mu}{1/\lambda+1/\mu} = \frac{a\mu+b\lambda}{\lambda+\mu}.$$

Q2: Solution

For a symmetric random telegraph process with $\lambda = \mu$, the number of switches N in a time interval τ follows the Poisson distribution

$$P_\tau(N = k) = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau}.$$

Assume that at $t = 0$ the system was in the state $s = +1$. Then, after τ seconds, the system will be in the state $s = +1$, if the number of switches in the interval of time τ is even and the system will be in the state $s = -1$, if the number of switches is odd.

The ACF can be written as follows

$$\begin{aligned} G(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T s(t)s(t+\tau)\, dt = \sum_{k=0}^{\infty} [P_\tau(N = 2k) - P_\tau(N = 2k+1)] \\ &= \left(1 - \frac{\lambda\tau}{1!} + \frac{(\lambda\tau)^2}{2!} - \frac{(\lambda\tau)^3}{3!} + \frac{(\lambda\tau)^4}{4!} - \dots\right) e^{-\lambda\tau} = e^{-\lambda\tau} e^{-\lambda\tau} = e^{-2\lambda\tau}. \end{aligned}$$

Q3: Solution

Arrival of busses at a bus stop is a renewal process with the pdf of the waiting times between two subsequent arrivals given by $f(t)$. If we arrive at the bus stop at a random (uniformly distributed) moment of time, then the time interval to the first arrival of the bus follows the distribution with the pdf

$$\psi(t) = \frac{\int_t^\infty f(t)\, dt}{\langle t \rangle},$$

where $\langle t \rangle = \int_0^\infty t f(t)\, dt$ is the average time between two subsequent arrivals of busses. For derivation of this formulae see (D.R. Cox, *The Renewal Theory*).

We want to show that the average waiting time for the next bus $\langle t_w \rangle = \int_0^\infty t\psi(t)\, dt$ is generally larger than $\langle t \rangle/2 = (1/2)\int_0^\infty t f(t)\, dt$. Thus, we have after using the integration by parts

$$\begin{aligned} \langle t_w \rangle &= \int_0^\infty t\psi(t)\, dt = (t^2\psi(t)/2)\Big|_0^\infty - \int_0^\infty \frac{t^2}{2}\psi(t)'\, dt \\ &= \int_0^\infty \frac{t^2}{2}\frac{f(t)}{\langle t \rangle}\, dt = \frac{1}{2}\frac{\langle t^2 \rangle}{\langle t \rangle}. \end{aligned}$$

Finally, using the positivity of the variance $\langle t^2 \rangle - \langle t \rangle^2 \geq 0$, we obtain

$$\langle t_w \rangle = \langle t \rangle \frac{1}{2}\frac{\langle t^2 \rangle}{\langle t \rangle^2} \geq \frac{1}{2}\langle t \rangle.$$