Stochastic Equations and Processes in physics and biology

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Lecture 2:

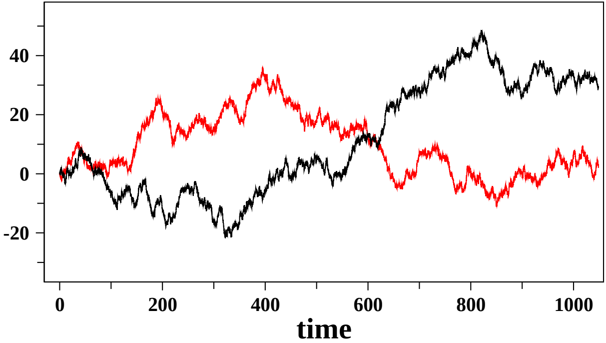
Stochastic Processes

Stochastic Process

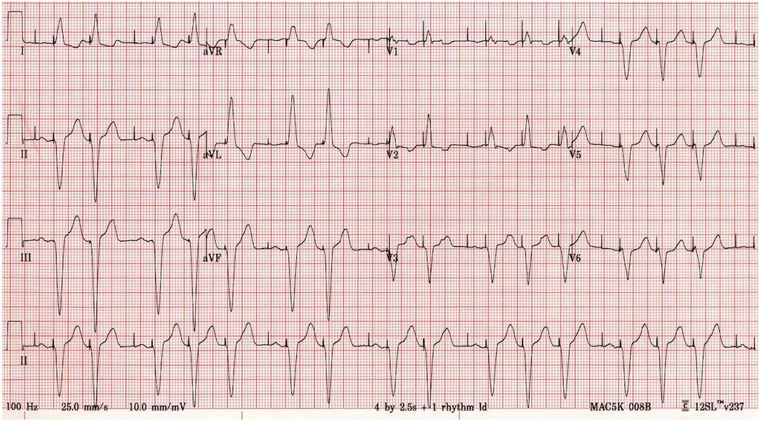
Definition

Any stochastic process is a probabilistic time series $x(t)$, where $x(t)$ is a time-dependent random variable

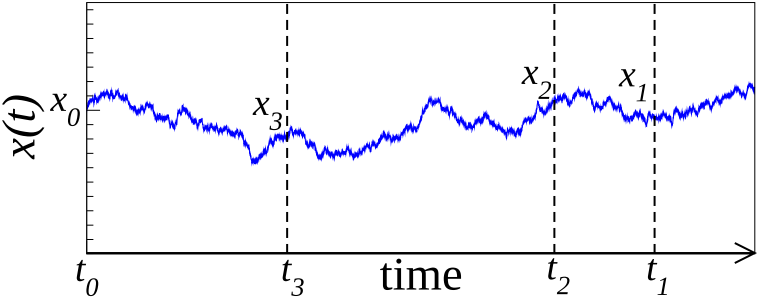
Coordinate of a Brownian particle *vs* time



Electrocardiogram (ECG)



Continuous stochastic process $x(t)$



Choose a series of points on the time line

$$(t_1 > t_2 > t_3 \dots) \text{ with } (x_1 = x(t_1), x_2 = x(t_2), x_3 = x(t_3) \dots)$$

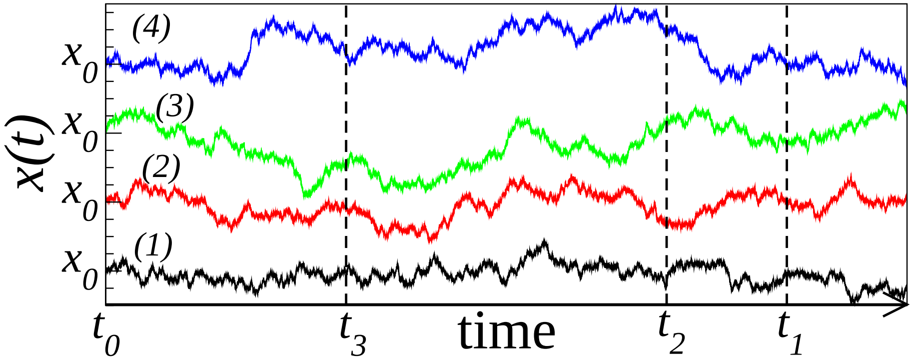
$x(t)$ **is completely described by the joint distribution**

$$p(x_1, t_1; x_2, t_2; x_3, t_3; \dots)$$

Ensemble of realizations

Role of initial conditions

For identical initial conditions $x(t=0) = x_0$, each realization (trajectory) of $x(t)$ is different!



Ensemble average

Averaging (integrating) over x

$$p(x_1, t_1) = \int_\Omega dx_2 \, p(x_1, t_1; x_2, t_2),$$

where Ω is the space of all possible values of x.

For $t_1 > t_2$ we define transitional probability to go from (2) to (1)

$$p(x_1, t_1 | x_2, t_2) = \frac{p(x_1, t_1; x_2, t_2)}{p(x_2, t_2)}$$

Time-dependent ensemble mean

$$\langle x(t) | x_0, t_0 \rangle = \int dx \, x \, p(x, t | x_0, t_0).$$

Autocorrelation function (ACF)

Definition

For process $x(t)$, defined on $(-\infty, \infty)$, we introduce the autocorrelation function

$$G(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt x(t) x(t+\tau),$$

ACF is an even function

$$G(-\tau) = G(\tau)$$

Autocorrelation function (ACF)

Proof of $G(-\tau) = G(\tau)$

$$\begin{aligned} G(-\tau) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt x(t) x(t-\tau) = \left\{ t - \tau = y \big|_{-\tau}^{T-\tau}, \;\; dt = dy \right\} \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\tau}^{T-\tau} dy x(\tau + y) x(y) \\ &= \lim_{T\to\infty} \frac{1}{T} \left[\int_{-\tau}^{0} (\dots) + \int_0^T (\dots) - \int_{T-\tau}^{T} (\dots) \right] = G(\tau) \end{aligned}$$

The last equality holds in the limit $T \to \infty$

ACF and the ensemble average

Non-stationary ACF

The conditional (non-stationary) ACF is determined as

$$\langle x(t)x(t')|x_0,t_0\rangle = \int dx\, dx'\, x\, x' p(x,t;x',t'|x_0,t_0)$$

Average over time vs ensemble average

For a general process $x(t)$

$$\langle x(t)x(t')|x_0,t_0\rangle \neq \lim_{T\to\infty}\frac{1}{T}\int_0^T dt x(t)x(t+\tau).$$

Markov processes

Absence of memory

For any $t_1 > t_2 > t_3$, the probability at time t_1 only conditionally depends on the state at time t_3, i.e.

$$p(x_1, t_1 | x_2, t_2; x_3, t_3) = p(x_1, t_1 | x_3, t_3).$$

Consequence

$$p(x_1, t_1; x_2, t_2 | x_3, t_3) = p(x_1, t_1 | x_2, t_2) p(x_2, t_2 | x_3, t_3).$$

Proof

Using the Markov property

$$
\begin{aligned}
p(x_1,t_1|x_2,t_2)p(x_2,t_2|x_3,t_3) &= p(x_1,t_1|x_2,t_2;x_3,t_3)p(x_2,t_2|x_3,t_3) \\
&= \frac{p(x_1,t_1;x_2,t_2;x_3,t_3)}{p(x_2,t_2;x_3,t_3)}\,\frac{p(x_2,t_2;x_3,t_3)}{p(x_3,t_3)} \\
&= \frac{p(x_1,t_1;x_2,t_2;x_3,t_3)}{p(x_3,t_3)} \\
&= p(x_1,t_1;x_2,t_2|x_3,t_3).
\end{aligned}
$$

The Chapman-Kolmogorov equation

Consider all possible ways to go from (3) **to** (1) **over** (2)

$$\begin{aligned} p(x_1,t_1|x_3,t_3) &= \int_\Omega dx_2\, p(x_1,t_1;x_2,t_2|x_3,t_3) \\ &= \int_\Omega dx_2\, \frac{p(x_1,t_1;x_2,t_2;x_3,t_3)}{p(x_3,t_3)} \end{aligned}$$

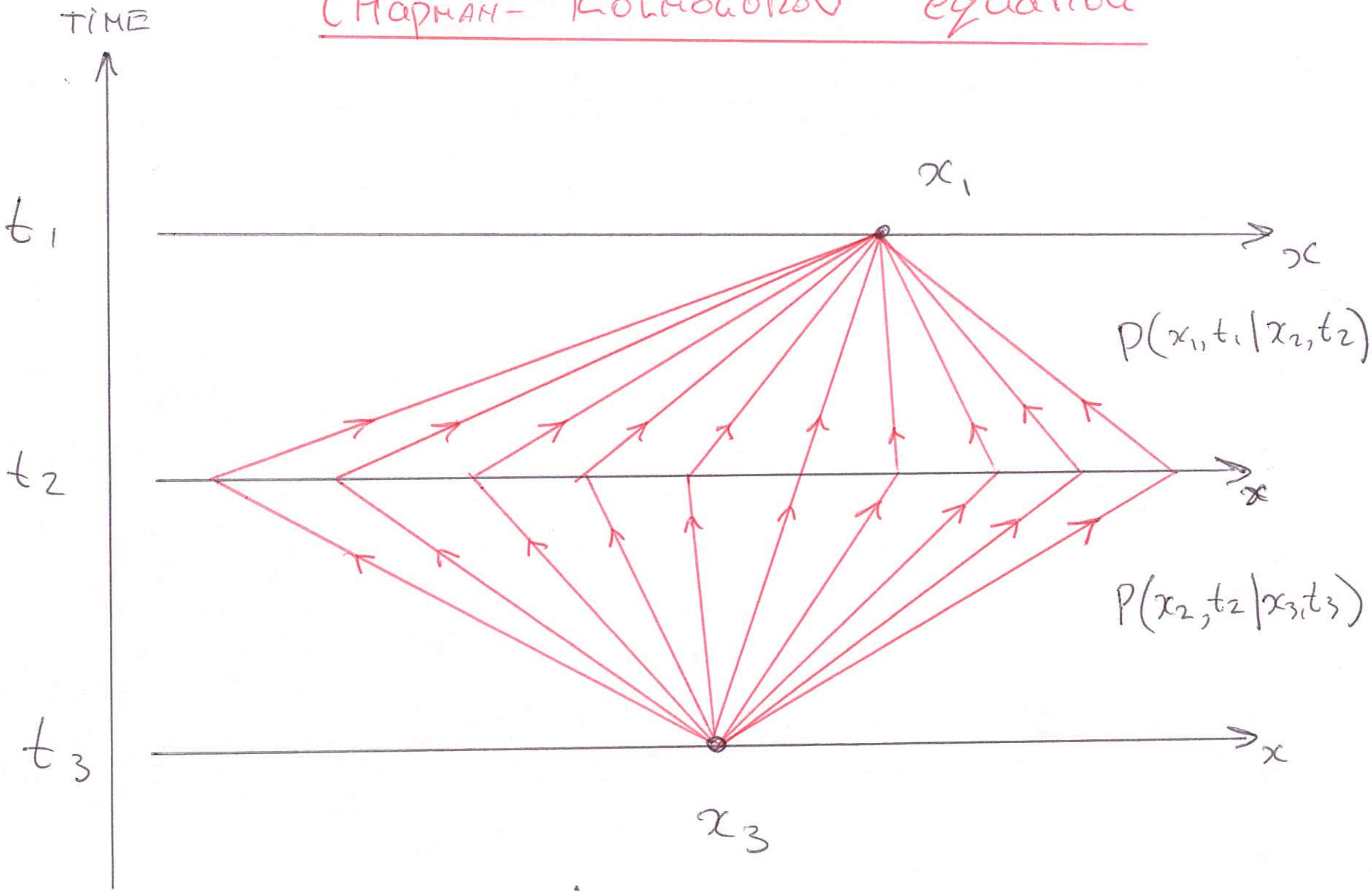
For Markovian process $x(t)$

$$\int_\Omega dx_2\, \frac{p(x_1,t_1;x_2,t_2;x_3,t_3)}{p(x_3,t_3)} = \int_\Omega dx_2\, p(x_1,t_1|x_2,t_2)p(x_2,t_2|x_3,t_3)$$

Chapman-Kolmogorov equation

$$p(x_1,t_1|x_3,t_3) = \int_\Omega dx_2\, p(x_1,t_1|x_2,t_2)p(x_2,t_2|x_3,t_3)$$

Chapman-Kolmogorov equation



$$P(x_1,t_1|x_3,t_3) = \int dx_2\, P(x_1,t_1|x_2,t_2)\, P(x_2,t_2|x_3,t_3)$$

Stationary processes

Definition

Process $x(t)$ is called stationary if for any $\epsilon > 0$, $x(t+\epsilon)$ has the same statistics as $x(t)$. Stationary process corresponds to a remote past:

$$t_0 \to -\infty.$$

Properties of a stationary process

$$\begin{aligned}
\lim_{t_0 \to -\infty} p(x,t|x_0,t_0) &= p_s(x) \\
\langle x(t)|x_0,t_0\rangle &= \int x\, p(x,t|x_0,t_0)\, dx = \int x p_s(x)\, dx \\
&= constant \\
\langle x(t)x(t')|x_0,t_0\rangle &= f(t-t').
\end{aligned}$$

ACF of a stationary processes

In the limit $t_0 \to -\infty$

$$
\begin{aligned}
\mathrm{ACF}(t,t')_s &= \lim_{t_0\to-\infty} \langle x(t)x(t')|x_0,t_0\rangle \\
&= \lim_{t_0\to-\infty} \int xx'dx\,dx'\,p(x,t;x',t'|x_0,t_0) \\
&= \lim_{t_0\to-\infty} \int xx'dx\,dx'\,p(x,t|x',t')p(x',t'|x_0,t_0) \\
&= \int xx'dx\,dx'\,p(x,t|x',t')p_s(x') \\
&= \int x'dx'\,\langle x(t)|x',t'\rangle p_s(x')
\end{aligned}
$$

Ergodic processes

Definition

For an ergodic process $x(t)$, the averaging over time is equivalent to the averaging over the ensemble.

Ergodicity vs Stationarity

Note that ergodicity is stronger than stationarity

Example:

$$x(t) = A, \;\; A \text{ is uniformly distributed in } [0;1]$$

Any realization is a straight line $x(t) = A_i$
A_i **are different for different realizations so that** $\langle x(t) \rangle = 0.5$
Average over time for a single realization $\int dt\, x(t) = A_i$

Random telegraph process

Kramers theory of chemical reactions: (H. A. Kramers, 1940)
Two reacting chemicals: X_1 **and** X_2

$$X_1 \rightleftharpoons X_2$$

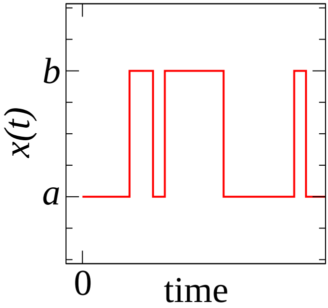
Associated bistable system $x(t) = (a, b)$ **with transition probabilities (reaction rates):**

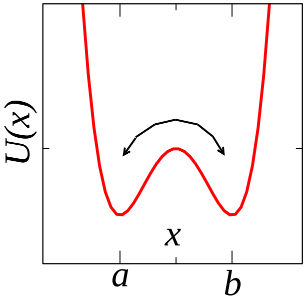
$$\lambda = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(x = b, t + \Delta t | x = a, t)$$

$$\mu = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(x = a, t + \Delta t | x = b, t)$$

Random telegraph process

Schematic representation and bistable systems





Other applications:

- in physics: spin systems and magnetism
- in finance: stoch market prices
- in biology: bistable neurons

Random telegraph process

Master equation (Chapman-Kolmogorov)

$$\begin{aligned}\partial_t P(a,t|x,t_0) &= -\lambda P(a,t|x,t_0) + \mu P(b,t|x,t_0)\\ \partial_t P(b,t|x,t_0) &= \lambda P(a,t|x,t_0) - \mu P(b,t|x,t_0)\end{aligned}$$

Conservation of probability at all times

$$\partial_t(P(a,t|x,t_0) + P(b,t|x,t_0)) = 0 \;\;\Rightarrow\;\; P(a,t|x,t_0) + P(b,t|x,t_0) = 1$$

Initial conditions

$$P(x',t_0|x_0,t_0) = \delta_{x',x_0} = \begin{cases} 0, & x' \neq x_0, \\ 1, & x' = x_0 \end{cases}$$

Random telegraph process

Characteristic eigenvalues γ

$$\begin{vmatrix} -\lambda-\gamma & \mu \\ \lambda & -\mu-\gamma \end{vmatrix} = 0$$

$$\gamma^2 + \gamma(\lambda+\mu) = 0$$

$$\gamma_1 = 0, \quad \gamma_2 = -(\lambda+\mu)$$

Corresponding eigenvectors

$$\boldsymbol{v}_{(\gamma=0)} = \left(C_1, \frac{\lambda}{\mu}C_1\right), \quad \boldsymbol{v}_{(\gamma=-(\lambda+\mu))} = (C_2, -C_2),$$

General solution

$$P(a,t|x_0,t_0) = C_1 + C_2 e^{-(\lambda+\mu)t},$$

$$P(b,t|x_0,t_0) = \frac{\lambda}{\mu}C_1 - C_2 e^{-(\lambda+\mu)t}$$

Random telegraph process

Solution in the compact form

$$
\begin{aligned}
P(a,t|x_0,t_0) &= \frac{\mu}{\lambda+\mu} + e^{-(\lambda+\mu)(t-t_0)}\left(\frac{\lambda}{\lambda+\mu}\delta_{a,x_0} - \frac{\mu}{\lambda+\mu}\delta_{b,x_0}\right), \\
P(b,t|x_0,t_0) &= \frac{\lambda}{\lambda+\mu} - e^{-(\lambda+\mu)(t-t_0)}\left(\frac{\lambda}{\lambda+\mu}\delta_{a,x_0} - \frac{\mu}{\lambda+\mu}\delta_{b,x_0}\right)
\end{aligned}
$$

Stationary distribution: $t_0 \to -\infty$

$$
P_s(a) = \frac{\mu}{\lambda+\mu}, \quad P_s(b) = \frac{\lambda}{\lambda+\mu}.
$$

Random telegraph process

Time-dependent ensemble average

$$
\begin{aligned}
\langle x(t)|x_0,t_0\rangle &= a\,P(a,t|x_0,t_0) + b\,P(b,t|x_0,t_0) \\
&= \frac{a\mu + b\lambda}{\mu+\lambda} + \exp\left[-(\lambda+\mu)(t-t_0)\right]\frac{(a-b)(\lambda\delta_{a,x_0} - \mu\delta_{b,x_0})}{\mu+\lambda}
\end{aligned}
$$

Note that

$$
\left(x_0 - \frac{a\mu + b\lambda}{\mu+\lambda}\right) = \frac{(a-b)(\lambda\delta_{a,x_0} - \mu\delta_{b,x_0})}{\mu+\lambda}
$$

Final result

$$
\langle x(t)|x_0,t_0\rangle = \frac{a\mu + b\lambda}{\mu+\lambda} + \exp\left[-(\lambda+\mu)(t-t_0)\right]\left(x_0 - \frac{a\mu + b\lambda}{\mu+\lambda}\right)
$$

Random telegraph process

Stationary average x_s

$$x_s = \lim_{t_0 \to -\infty} \langle x(t) | x_0, t_0 \rangle = \frac{a\mu + b\lambda}{\mu + \lambda}.$$

Stationary variance $\mathrm{Var}(x)_s = \langle x^2 \rangle_s - x_s^2$

$$\begin{aligned}
\mathrm{Var}(x)_s &= \frac{a^2\mu}{\lambda + \mu} + \frac{b^2\lambda}{\lambda + \mu} - \frac{(a\mu + b\lambda)^2}{(\lambda + \mu)^2} \\
&= \frac{a^2\mu(\lambda + \mu) + b^2\lambda(\lambda + \mu) - (a^2\mu^2 + b^2\lambda^2 - 2ab\mu\lambda)}{(\lambda + \mu)^2} \\
&= \frac{(a - b)^2\mu\lambda}{(\lambda + \mu)^2}
\end{aligned}$$

Random telegraph process

Stationary ACF

$$
\begin{aligned}
\mathrm{ACF}(t,t')_s &= \sum_{(x,x'=a,b)} x\,x' P(x,t|x',t') P_s(x') \\
&= \sum_{(x'=a,b)} x' P_s(x') \sum_{(x=a,b)} x\,P(x,t|x',t') \\
&= \sum_{(x'=a,b)} x' \langle x(t)|x',t'\rangle P_s(x') \\
&= a\langle x(t)|a,t'\rangle P_s(a) + b\langle x(t)|b,t'\rangle P_s(b) \\
&= \left(\frac{a\mu + b\lambda}{\mu+\lambda}\right)^2 + \frac{(a-b)^2\mu\lambda}{(\mu+\lambda)^2} \exp\left[-(\lambda+\mu)(t-t')\right] \\
&= x_s^2 + \mathrm{Var}(x)_s \exp\left[-(\lambda+\mu)(t-t')\right]
\end{aligned}
$$

Random telegraph process

Process centered about mean value $x(t) - x_s$

$$\langle x(t) - x_s | x_0, t_0 \rangle = \exp\left[-(\lambda + \mu)(t - t_0)\right](x_0 - x_s)$$

Stationary ACF of the centered process

$$
\begin{aligned}
G(t,t') &= \langle (x(t)-x_s)(x(t')-x_s)|x_0,t_0\rangle \\
&= \int dx\, dx'(x-x_s)(x'-x_s)p(x,t;x',t'|x_0,t_0) \\
&= \int dx\, dx'(x\,x'-x_s x-x_s x'+x_s^2)p(x,t;x',t'|x_0,t_0) \\
&= \langle x(t)x(t')|x_0,t_0\rangle \\
&- x_s\int dx\, dx'\, x' p(x,t;x',t'|x_0,t_0) \\
&- x_s\int dx\, dx'\, x p(x,t;x',t'|x_0,t_0) \\
&+ x_s^2
\end{aligned}
$$

Random telegraph process

Note that

$$\begin{aligned}
\int dx\, dx'\, x' p(x,t;x',t'|x_0,t_0) &= \int dx'\, x' p(x',t'|x_0,t_0) \\
&= \langle x(t')|x_0,t_0\rangle \\
\int dx\, dx'\, x p(x,t;x',t'|x_0,t_0) &= \int dx\, x p(x,t|x_0,t_0) \\
&= \langle x(t)|x_0,t_0\rangle
\end{aligned}$$

in the limit $t_0 \to -\infty$

$$\langle x(t)|x_0,t_0\rangle = \langle x(t')|x_0,t_0\rangle = x_s$$

Random telegraph process

Stationary ACF of the centered process

$$
\begin{aligned}
G(t, t') &= \mathrm{ACF}(t, t')_s - \langle x \rangle_s^2 \\
&= \mathrm{Var}(x)_s \exp\left[-(\lambda + \mu)(t - t')\right]
\end{aligned}
$$

Random telegraph process

Survival probability

Given that at time $t = t_0$ the system was in state $x = a$, what is the probability $P(a,t|a,t_0)$ for the system to stay in the same state at time $t > t_0$?

$$
\begin{aligned}
\partial_t P(a,t|a,t_0) &= -\lambda P(a,t|a,t_0), \quad P(a,t=t_0|a,t_0) = 1 \\
\partial_t P(b,t|b,t_0) &= -\mu P(b,t|b,t_0), \quad P(b,t=t_0|b,t_0) = 1
\end{aligned}
$$

Survival probabilities for a random telegraph process

$$
P(a,t|a,t_0) = e^{-\lambda(t-t_0)}, \qquad P(b,t|b,t_0) = e^{-\mu(t-t_0)}.
$$

Residence times

$t_a \ldots$ time spent in state a
$t_b \ldots$ time spent in state b

cdf of the residence times

$$\mathrm{cdf}_a(t) = \Pr(t_a \le t) = 1 - \Pr(t_a \ge t) = 1 - P(a,t|a,t_0) = 1 - e^{-\lambda(t-t_0)}$$

$$\mathrm{cdf}_b(t) = \Pr(t_b \le t) = 1 - \Pr(t_b \ge t) = 1 - P(b,t|b,t_0) = 1 - e^{-\mu(t-t_0)}$$

pdf and the average residence times

$$\mathrm{pdf}_a(t) = \mathrm{cdf}_a(t)' = \lambda e^{-\lambda(t-t_0)}, \;\; \langle t_a \rangle = \lambda^{-1}$$

$$\mathrm{pdf}_b(t) = \mathrm{cdf}_b(t)' = \mu e^{-\mu(t-t_0)}, \;\; \langle t_b \rangle = \mu^{-1}$$

Relation of the random telegraph process to the Poisson distribution

Exponential *vs* Poisson

If the distribution of the residence times is exponential with the parameter λ, then that distribution of the number of switches N in any interval τ follows the Poisson distribution

$$P(N = k) = \frac{(\tau\lambda)^k}{k!} e^{-\tau\lambda}.$$

Expected value and variance

$$E(N) = \tau\lambda, \qquad \mathrm{Var}(N) = \tau\lambda$$

Examples of the Poisson distribution

- Radioactivity (number of decays in a given time interval)
- Retail markets (number of customers arriving at a shop in a given time interval, or the number of purchases per day, per hour or per minute)
- Telecommunication (number of telephone calls arriving per given time interval)

Renewal process

Sequence of random variables s_i with identical distribution

$$(s_1, s_2, s_3, \dots)$$

General theory

- David R. Cox *Renewal Theory*
- Igor Goychuk and Peter Hänggi, *Theory of non-Markovian stochastic resonance*, Phys Rev. E **69**, 021104 (2004)

Firing neuron as a renewal process