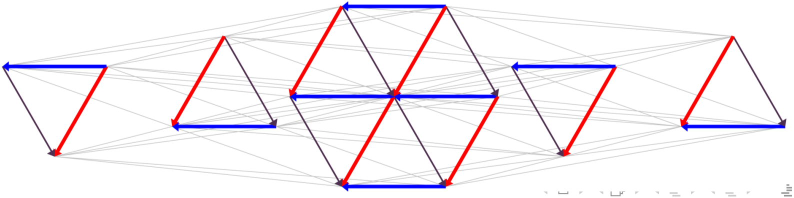
(Kac–Moody) Chevalley groups and Lie algebras with built–in structure constants

Lecture 1

Lisa Carbone

Rutgers University



Slides will be posted at:

http://sites.math.rutgers.edu/∼carbonel/

Video will be posted at:

http://research.amsi.org.au/ace-short-courses/

or email me:

lisa.carbone@rutgers.edu

Topics

(1) Overview and introductory comments

(2) Lie algebras: finite and infinite dimensional

(3) Weights, representations and universal enveloping algebra

(4) (Kac–Moody) Chevalley groups

(5) Generators and relations and (Kac–Moody) Groups over $\mathbb{Z}$

(6) Structure constants for Kac–Moody algebras and Chevalley groups

Some references

Classical theory:

Humphreys, J. E. *Introduction to Lie algebras and representation theory*. GTM, 9. Springer-Verlag, New York-Berlin, (1978).

V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge, UK: Univ. Pr. (1990) 400 p.

Moody, R. V. and Pianzola, A. *Lie algebras with triangular decompositions*, Canadian Mathematical Society. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, (1995).

Steinberg, R. *Lectures on Chevalley groups*, Notes prepared by John Faulkner and Robert Wilson, Yale University, (1967)

New results:

Carbone, L., Coulson, B., Kanade, S., McRae, R. H. and Murray, S. H. *Structure constants for Kac–Moody algebras*, In preparation (2017)

Carbone, L., Kownacki, M., Murray, S. H. and Srinivasan, S. *Structure constants for rank 2 Kac–Moody algebras*, Preprint, (2017)

Carbone, L. and Liu, D. *Infinite dimensional Chevalley groups and Kac–Moody groups over $\mathbb{Z}$*, Preprint (2017)

(1) Overview and introductory comments

Cartan-Killing classification of finite dimensional simple Lie algebras:

Classical types:

$$A_n, \; n \geq 1: \quad \mathfrak{sl}_{n+1}$$

$$B_n, \; n \geq 1: \quad \mathfrak{so}_{2n+1}$$

$$C_n, \; n \geq 1: \quad \mathfrak{sp}_{2n}$$

$$D_n, \; n \geq 3: \quad \mathfrak{so}_{2n}$$

exceptional types:

$$\mathfrak{e}_6, \; \mathfrak{e}_7, \; \mathfrak{e}_8, \; \mathfrak{f}_4, \; \mathfrak{g}_2$$

corresponding to Lie groups

$$SL_{n+1}(\mathbb{C}), \; SO_{2n+1}(\mathbb{C}), \; Sp_{2n}(\mathbb{C}), \; SO_{2n}(\mathbb{C})$$
$$E_6, \; E_7, \; E_8, \; F_4, \; G_2.$$

Most of the group constructions, particularly the exceptional ones, are *type dependent*.

The role of Chevalley groups

By 1950, a unified method for constructing simple Lie groups was still missing.

The subject was clarified by the introduction of the theory of algebraic groups, and the work of Chevalley in 1955, which gave an alternate construction of the simple algebraic groups as automorphisms $Aut(\mathfrak{g})$ of the underlying Lie algebra or a representation space.

This approach uses external data but has significant advantages, leading to:

- A unified description of groups over arbitrary fields and over $\mathbb{Z}$.
- Explicit determination of structure constants.
- Generators and defining relations for the Chevalley group.

Our objective will be to study Chevalley's construction and apply it to the infinite dimensional case: namely to associate Chevalley-type groups to infinite dimensional Lie algebras known as Kac–Moody algebras.

Some fundamental questions

The finite dimensional classical groups

$$SL_{n+1}(\mathbb{C}),\ SO_{2n+1}(\mathbb{C}),\ Sp_{2n}(\mathbb{C}),\ SO_{2n}(\mathbb{C})$$

are naturally defined in terms of matrices.

How do we define the exceptional groups E_6, E_7, E_8, F_4, G_2?

There are different forms and type-dependent constructions.

How do we define Lie groups over $\mathbb{Z}$?

The classical groups can be described as groups of matrices with $\mathbb{Z}$-entries.

For exceptional Lie groups and infinite dimensional Kac–Moody groups, another method is required to define these groups over $\mathbb{Z}$.

Structure constants

Let $\mathfrak{g}$ be a Lie algebra or Kac–Moody algebra. As a vector space $\mathfrak{g}$ over $\mathbb{C}$ with a bilinear operation

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

the structure constants of $\mathfrak{g}$ are the constants that occur in the evaluation of the Lie bracket in terms of a choice of basis for $\mathfrak{g}$. If $\mathfrak{g}$ has basis $\{x_i\}_{i=1,\ldots}$ and Lie bracket $[\cdot,\cdot]$ defined by

$$[x_i, x_j] = \sum_k n_{ijk} x_k.$$

The elements $n_{ijk} \in \mathbb{C}$ are the *structure constants* of $\mathfrak{g}$ and they depend on the choice of basis for $\mathfrak{g}$.

How do we determine the $n_{ijk} \in \mathbb{C}$? Chevalley answered this question in terms of the roots of the Lie algebra, but only up to a sign (± 1).

We will discuss recent work on determining the signs of structure constants for Kac–Moody algebras.

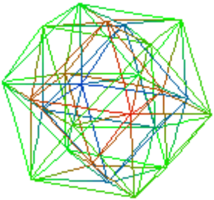
Our approach

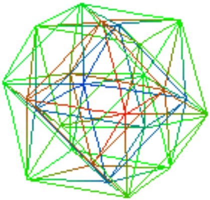
We will study Chevalley's theory, giving a unified construction of classical and exceptional groups, and their structure constants, over arbitrary fields, and also over $\mathbb{Z}$.

This will extend to a general construction of infinite dimensional Kac–Moody groups as Chevalley groups.

These are groups associated to infinite dimensional Kac–Moody algebras which are the most natural extension to infinite dimensions of finite dimensional simple Lie algebras.

We start with a definition of Lie algebras that includes both the finite dimensional and the infinite dimensional case.





(2) Lie algebras: finite and infinite dimensional

The data needed to construct a Lie algebra includes the following.

Let $A = (a_{ij})_{i,j \in I}$ be a *generalized Cartan matrix*. The entries of A satisfy the conditions
$a_{ij} \in \mathbb{Z}, i, j \in I,$
$a_{ii} = 2, i \in I,$
$a_{ij} \leq 0$ if $i \neq j$, and
$a_{ij} = 0 \iff a_{ji} = 0.$ We will assume throughout that A is

symmetrizable: there exist positive rational numbers $q_1, \ldots, q_\ell$, such that the matrix DA is symmetric, where $D = diag(q_1, \ldots, q_\ell)$.

For example, the Cartan matrix for G_2 can be decomposed:

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}.$$

The diagonal entries give the relative lengths of the simple roots.

Lie algebras: finite and infinite dimensional

Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix.

We say that A is of *finite type* if A is positive definite. In this case, $det(A) > 0$.

We say that A is of *affine type* if A is positive-semidefinite, but not positive-definite. In this case, $det(A) = 0$.

If A is not of finite or affine type, we say that A has *indefinite type*. In this case, $det(A) < 0$.

These are 3 mutually exclusive cases.

Among the indefinite types, we are interested in the case that A is of *hyperbolic type*: that is, A is neither of finite nor affine type, but every proper, indecomposable submatrix is either of finite or of affine type.

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} A_1^{(1)} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & A_2 \end{pmatrix}$$

Examples: Generalized Cartan matrices

The possible generalized Cartan matrices have been classified ([K], Ch4). Here is the classification in rank 2:

Rank 2 finite type: $det(A) > 0$

$$A_1 \times A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \; A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \; G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Rank 2 affine Kac–Moody type: $det(A) = 0$

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \; A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix},$$

Rank 2 hyperbolic Kac–Moody type: $det(A) < 0$

$$H(m) = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}_{m \in \mathbb{Z}_{\geq 3}}, \; H(a,b) = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}_{ab > 4}$$

The Dynkin diagram of a generalized Cartan matrix

The *Dynkin diagram* $\mathcal{D}$ of a generalized Cartan matrix $A = (a_{ij})$ is the graph with one node for each row (or column) of A.

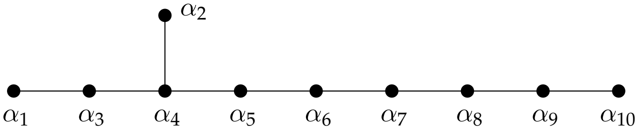
If $i \neq j$ and $a_{ij}a_{ji} \leq 4$ then we connect i and j with $\max(|a_{ij}|, |a_{ji}|)$ edges together with an arrow towards i if $|a_{ij}| > 1$.

If $a_{ij}a_{ji} > 4$ we draw a bold face line labeled with the ordered pair $(|a_{ij}|, |a_{ji}|)$.

The classification of Dynkin diagrams of finite type is well known ([Hu], Ch III).

The affine Dynkin diagrams have been classified ([K], Ch 4).

There are 238 hyperbolic Dynkin diagrams in ranks 3-10, 142 of which are symmetrizable ([CCCMNNP]).



Lie algebras: generators

Let $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ denote the natural nondegenerate bilinear pairing between a vector space $\mathfrak{h}$ and its dual.

Given

– a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, $I = \{1, \ldots, \ell\}$,

– a finite dimensional vector space $\mathfrak{h}$ (Cartan subalgebra) with $dim(\mathfrak{h}) = 2\ell - rank(A)$,

– a choice of *simple roots* Π and *simple coroots* $\Pi^\vee$

$$\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathfrak{h}^*, \ \Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$$

such that Π and $\Pi^\vee$ are linearly independent and such that

$$\langle \alpha_j, \alpha_i^\vee \rangle = \alpha_j(\alpha_i^\vee) = a_{ij},$$

$i \in I$, we may associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$, with generators

$$\mathfrak{h}, \ (e_i)_{i \in I}, \ (f_i)_{i \in I}.$$

Lie algebras: generators and defining relations

The Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ generated by

$$\mathfrak{h},\ (e_i)_{i\in I},\ (f_i)_{i\in I}.$$

is subject to defining relations ([Hu], Ch IV, [K], Ch 9, [M]):

$$[\mathfrak{h},\mathfrak{h}] = 0,$$
$$[h,e_i] = \langle \alpha_i, h\rangle e_i,\ h \in \mathfrak{h},$$
$$[h,f_i] = -\langle \alpha_i, h\rangle f_i,\ h \in \mathfrak{h},$$
$$[e_i,f_i] = \alpha_i^\vee,$$
$$[e_i,f_j] = 0,\ i \neq j,$$
$$(ad\ e_i)^{-a_{ij}+1}(e_j) = 0,\ i \neq j,$$
$$(ad\ f_i)^{-a_{ij}+1}(f_j) = 0,\ i \neq j, \text{ where } (ad(x))(y) = [x,y].$$

The last two relations, due to Serre, restrict the growth of commutators.

The algebra $\mathfrak{g} = \mathfrak{g}(A)$ is infinite dimensional if A is not positive definite.

The Serre relations

The relations

$(ad\ e_i)^{-a_{ij}+1}(e_j) = 0,\ i \neq j,$
$(ad\ f_i)^{-a_{ij}+1}(f_j) = 0,\ i \neq j,$

where $(ad(x))(y) = [x, y]$, are known as the *Serre relations*.

For example, if $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, the Serre relations give

$$[e_1, [e_1, [e_1, e_2]]] = 0, \quad \text{but } [e_1, [e_1, e_2]] \neq 0.$$

Here the nonzero double commutator causes $\mathfrak{g}(A)$ to be infinite dimensional.

If $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$, the Serre relations give

$$[e_1[e_1, [e_1, [e_1, e_2]]]] = 0, \quad \text{but } [e_1[e_1, [e_1, e_2]]] \neq 0,$$

which leads to exponential growth of the nonzero commutators.

Example: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

Let A be the Cartan matrix $A = (2)$. Let $\mathfrak{h}$ be a 1 dimensional vector space. Choose

$$\alpha \in \mathfrak{h}^* \text{ and } \alpha^\vee \in \mathfrak{h}$$

such that

$$\langle \alpha, \alpha^\vee \rangle = \alpha(\alpha^\vee) = 2.$$

We may associate a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, generated by symbols

$$h,\ e,\ f$$

such that

$$[h,h] = 0,\ [e,f] = h,\ [h,e] = 2e \text{ and } [h,f] = -2f.$$

The Cartan subalgebra $\mathfrak{h}$ acts on $\mathfrak{g}$ by the automorphism

$$\begin{aligned} ad_{\mathfrak{h}} : \mathfrak{g} &\to \mathfrak{g} \\ x &\mapsto [h,x] \end{aligned}$$

for $h \in \mathfrak{h}$ and this action yields the decomposition

$$\mathfrak{g} = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f.$$

Example: the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

The 3–dimensional Lie algebra $\mathfrak{g}$ can be realized by choosing

$$\mathfrak{h} = \mathbb{C}h \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{C} \right\},$$

$$\alpha \in \mathfrak{h}^* \text{ such that } \alpha^\vee = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}),$$

the Lie algebra of 2×2 matrices of trace 0 over $\mathbb{C}$.

Lie algebras: general construction

In general, the Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ decomposes into a direct sum of root spaces ([K], Theorem 1.2)

$$\mathfrak{g} = \mathfrak{n}_- \ \oplus \ \mathfrak{h} \ \oplus \ \mathfrak{n}_+$$

under the simultaneous adjoint action of $\mathfrak{h}$, where

$$\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha, \ \ \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha.$$

The roots $\Delta := \Delta_+ \sqcup \Delta_- \ \subset \ \mathfrak{h}^*$ are the eigenvalues, and the root spaces

$$\mathfrak{g}_\alpha \ = \ \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \ h \in \mathfrak{h}\}$$

are the corresponding eigenspaces, satisfying $dim(\mathfrak{g}_\alpha) < \infty$.

If $\mathfrak{g} = \mathfrak{g}(A)$ is finite dimensional, then $dim(\mathfrak{g}_\alpha) = 1$ for each root α.

If A is symmetrizable, the algebra $\mathfrak{g}$ admits a well-defined non-degenerate symmetric bilinear form $(\cdot \mid \cdot)$ which plays the role of 'squared length' of a root when restricted to $\mathfrak{h}^*$ ([K], Theorem 2.2) and corresponds to the Killing form in finite dimensions ([Hu], Ch II).

Root systems and Weyl group

For each simple root α_i, $i \in I$, we define the simple root reflection

$$w_i(\alpha_j) = \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i = \alpha_j - a_{ij}\alpha_i.$$

It follows that $w_i(\alpha_i) = -\alpha_i$. The w_i generate a subgroup

$$W = W(A) \subseteq Aut(\mathfrak{h}^*),$$

called the *Weyl group* of A, and $(\cdot \mid \cdot)\mid_{\mathfrak{h}^*}$ is W-invariant.

A root $\alpha \in \Delta$ is called a *real root* if there exists $w \in W$ such that $w\alpha$ is a simple root. Otherwise α is called *imaginary*.

We denote by Δ^{re} the real roots, Δ^{im} the imaginary roots.

It follows that $\Delta^{re} = W\Pi$. The group W acts on Δ^{re} and Δ^{im}.

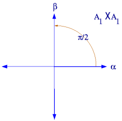
If A has finite type, then $|\Delta| < \infty$, $\Delta = \Delta^{re}$ and $\Delta^{im} = \varnothing$.

Otherwise $|\Delta| = \infty$, $|\Delta^{re}| = \infty$, $|\Delta^{im}| = \infty$.

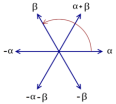
If $\alpha \in \Delta^{re}$, then $dim(\mathfrak{g}_\alpha) = 1$, otherwise $dim(\mathfrak{g}_\alpha) \geq 1$ and $dim(\mathfrak{g}_\alpha)$ grows exponentially if $\mathfrak{g}$ is hyperbolic.

Rank 2 finite root systems

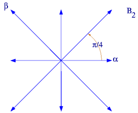
1. $A_1 \times A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$



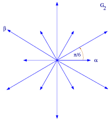
2. $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$



3. $B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

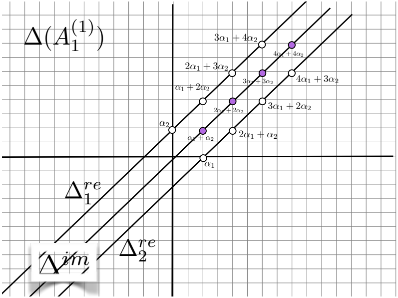


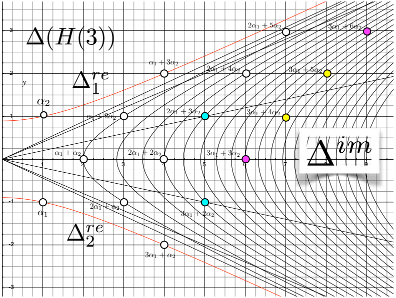
4. $G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$



Root systems of $A_1^{(1)}$ and $H(3)$ in $\mathfrak{h}_{\mathbb{R}}$

For both $A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $H(3) = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$, we may take $\mathfrak{h}_{\mathbb{R}} = \mathbb{R}^{1,1}$, where $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.





$\alpha_1 + 2\alpha_2 = w_2\alpha_1,\ 2\alpha_1 + \alpha_2 = w_1\alpha_2 \ldots \quad \alpha_1 + 3\alpha_2 = w_2\alpha_1,\ 3\alpha_1 + \alpha_2 = w_1\alpha_2 \ldots$

$$\alpha \in \Delta^{re} \iff (\alpha, \alpha) > 0$$
$$\alpha \in \Delta^{im} \iff (\alpha, \alpha) \leq 0$$

Root lattice and weight lattice

The roots Δ lie on a lattice in $\mathfrak{h}^*$ called the *root lattice*, denoted Q. We have $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell$, a free $\mathbb{Z}$-module.

Every $\alpha \in \Delta$ has an expression in Q of the form

$$\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$$

where the k_i are either all ≥ 0, in which case α is called *positive*, or all ≤ 0, in which case α is called *negative*.

The positive roots are denoted Δ_+, the negative roots Δ_-.

The *weight lattice* P in $\mathfrak{h}^*$ is:

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z},\ i = 1, \ldots \ell\}.$$

The *dominant integral weights* are

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0},\ i = 1, \ldots \ell\}.$$

Root lattice and weight lattice

The weight lattice P contains a basis of *fundamental weights* $\{\omega_1, \ldots, \omega_\ell\} \subset \mathfrak{h}^*$ such that

$$\langle \omega_i, \alpha_j^\vee \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & i \neq j. \end{cases}$$

We write

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z},\ i = 1, \ldots \ell\} = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_\ell.$$

Since $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij} \in \mathbb{Z}$, $i, j = 1, \ldots \ell$, we have $\alpha_i \in P$ so roots are weights and thus $Q \leq P$.

The index of the root lattice Q as a subgroup of the weight lattice P is finite and is given by $|det(A)|$ where the generalized Cartan matrix A.

ROOT LATTICE AND WEIGHT LATTICE OF TYPE A_2