(Kac–Moody) Chevalley groups and Lie algebras with built–in structure constants

Lecture 2

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TOPICS

(1) Overview and introductory comments

(2) Lie algebras: finite and infinite dimensional

(3) Weights, representations and universal enveloping algebra

(4) (Kac–Moody) Chevalley groups

(5) Generators and relations and (Kac–Moody) Groups over $\mathbb{Z}$

(6) Structure constants for Kac–Moody algebras and Chevalley groups

Today we will use a class of representations of (Kac–Moody) Lie algebras to construct groups known as (Kac–Moody) Chevalley groups.

We will construct two forms of (Kac–Moody) Chevalley groups: the adjoint form using the adjoint representation and a simply connected form using a more general representation known as a highest weight module.

Last time

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix:

$a_{ij} \in \mathbb{Z}, i, j \in I,$
$a_{ii} = 2, i \in I,$
$a_{ij} \leq 0$ if $i \neq j$, and
$a_{ij} = 0 \iff a_{ji} = 0.$

Let $\mathfrak{h}$ be a finite dimensional vector space (Cartan subalgebra).

Let $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \longrightarrow \mathfrak{h}$ denote the natural nondegenerate bilinear pairing between $\mathfrak{h}$ and its dual.

Let Π and $\Pi^\vee$ be a choice of *simple roots* and *simple coroots*

$$\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathfrak{h}^*, \ \Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$$

such that Π and $\Pi^\vee$ are linearly independent and such that

$$\langle \alpha_j, \alpha_i^\vee \rangle = \alpha_j(\alpha_i^\vee) = a_{ij}.$$

We associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$, with generators

$$\mathfrak{h}, \ (e_i)_{i\in I}, \ (f_i)_{i\in I}.$$

Lie algebras: generators and defining relations

The Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ generated by

$$\mathfrak{h},\ (e_i)_{i\in I},\ (f_i)_{i\in I}.$$

is subject to defining relations ([Hu], Ch IV, [K], Ch 9, [M]):

$$[\mathfrak{h},\mathfrak{h}] = 0,$$
$$[h,e_i] = \langle \alpha_i, h\rangle e_i, h \in \mathfrak{h},$$
$$[h,f_i] = -\langle \alpha_i, h\rangle f_i, h \in \mathfrak{h},$$
$$[e_i,f_i] = \alpha_i^\vee,$$
$$[e_i,f_j] = 0,\ i \neq j,$$
$$(ad\ e_i)^{-a_{ij}+1}(e_j) = 0,\ i \neq j,$$
$$(ad\ f_i)^{-a_{ij}+1}(f_j) = 0,\ i \neq j, \text{ where } (ad(x))(y) = [x,y].$$

The algebra $\mathfrak{g} = \mathfrak{g}(A)$ is infinite dimensional if A is not positive definite and is called a *Kac–Moody algebra*.

LIE ALGEBRAS: GENERAL CONSTRUCTION

The algebra $\mathfrak{g}$ decomposes into a direct sum of root spaces

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

under the simultaneous adjoint action of $\mathfrak{h}$, where

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha.$$

The roots $\Delta := \Delta_+ \sqcup \Delta_- \subset \mathfrak{h}^*$ are the eigenvalues, and the root spaces

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x,\ h \in \mathfrak{h}\}$$

are the corresponding eigenspaces, satisfying $dim(\mathfrak{g}_\alpha) < \infty$.

If $\mathfrak{g} = \mathfrak{g}(A)$ is finite dimensional, then $dim(\mathfrak{g}_\alpha) = 1$ for each root α.

For each simple root α_i, $i \in I$, we define the simple root reflection

$$w_i(\alpha_j) = \alpha_j - \alpha_j(\alpha_i^\vee)\alpha_i = \alpha_j - a_{ij}\alpha_i.$$

The w_i generate a subgroup

$$W = W(A) \subseteq Aut(\mathfrak{h}^*),$$

called the *Weyl group* of A.

Root lattice and weight lattice

The roots Δ of $\mathfrak{g}$ lie on a lattice in $\mathfrak{h}^*$ called the *root lattice*, denoted Q. We have $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_\ell$.

The *weight lattice* P in $\mathfrak{h}^*$ is:

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z},\ i = 1, \ldots \ell\} = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_\ell.$$

The *dominant integral weights* are

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0},\ i = 1, \ldots \ell\}.$$

Since $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij} \in \mathbb{Z}$, $i, j = 1, \ldots \ell$, we have $\alpha_i \in P$ so roots are weights and thus $Q \leq P$.

The index of the root lattice Q as a subgroup of the weight lattice P is finite and is given by $|det(A)|$ where the generalized Cartan matrix A.

For example, the fundamental weight of $\mathfrak{sl}_2$ is

$$\omega = A^{-1}\alpha = \frac{1}{2}\alpha$$

where $A = (2)$.

(3) Weights, representations and universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra or Kac–Moody algebra.

Some of the weights $\omega \in P$ are related to the representations of $\mathfrak{g}$.

Let V be a $\mathfrak{g}$–module with defining representation $\rho : \mathfrak{g} \to End(V)$.

The *weight space* of V with weight $\mu \in P$ is

$$V_\mu = \{v \in V \mid x \cdot v = \mu(x)v \text{ for all } x \in \mathfrak{h}\}.$$

The set of weights of the representation V is

$$wts(V) = \{\mu \in P \mid V_\mu \neq 0\}.$$

If $\mu_1, \mu_2, \ldots, \mu_n$ are weights of a representation V, then the lattice L_V generated by these weights is

$$L_V = \mathbb{Z}\mu_1 \oplus \mathbb{Z}\mu_2 \oplus \cdots \oplus \mathbb{Z}\mu_n = \{\textstyle\sum_{i=1}^n a_i\mu_i \mid a_i \in \mathbb{Z}_{\geq 0}\}$$

which is a subgroup of P.

Integrable representations

A $\mathfrak{g}$–module V is called *integrable* if it is diagonalizable, that is, V can be written as a direct sum of its weight spaces:

$$V = \bigoplus_{\lambda \in wts(V)} V_\lambda$$

and if the e_i and f_i act *locally nilpotently* on V. That is, for each $v \in V$,

$$e_i^{n_i} \cdot v = f_i^{n_i} \cdot v = 0$$

for all $i \in I$ and for some $n_i > 0$.

If $\mathfrak{g}$ is finite dimensional, then every finite dimensional representation V of $\mathfrak{g}$ is integrable.

The adjoint representation of a Lie algebra or Kac–Moody algebra $\mathfrak{g}$

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$x \mapsto ad\ x$$

is integrable.

(Kac–Moody) Chevalley groups of adjoint type

Let $\mathfrak{g}$ be a simple Lie algebra or Kac–Moody algebra with generators e_i and f_i, $i \in I$. Let Δ be the root system of $\mathfrak{g}$. Let

$$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

$$x \mapsto ad\ x$$

be the adjoint representation. Then

$$(ad\ x)(y) = [x, y].$$

Let $\alpha \in \Delta$. Let $s, t \in \mathbb{C}$ and set

$$exp(s \cdot ad(e_i)) = I + s \cdot ad(e_i) + \frac{s^2}{2} \cdot (ad(e_i))^2 + \dots$$

$$exp(t \cdot ad(f_i)) = I + t \cdot ad(f_i) + \frac{t^2}{2} \cdot (ad(f_i))^2 + \dots$$

Since $ad(\mathfrak{g})$ is locally nilpotent, these are well defined elements of $GL(\mathfrak{g})$.

(Kac–Moody) Chevalley groups of adjoint type

The (Kac–Moody) Chevalley group of adjoint type is defined as

$$G_{ad} = \langle exp(s \cdot ad(e_i)), exp(t \cdot ad(f_i)) \mid i \in I, \alpha \in \Delta,\ s, t \in \mathbb{C} \rangle < GL(\mathfrak{g}).$$

The adjoint representation $ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ gives rise to a representation

$$Ad : G_{ad} \longrightarrow GL(\mathfrak{g})$$

such that

$$exp(ad(x)) = Ad(exp(x))$$

for all $x \in \mathfrak{g}$. It is routine to verify that for all $g \in G$, $x \in \mathfrak{g}$,

$$exp(Ad\, g)(x)) \ = \ g\, exp(x)\, g^{-1}.$$

Adjoint Chevalley group of $\mathfrak{sl}_2$

For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the matrices for $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ acting on the basis $\{e, f, h\}$ for $\mathfrak{sl}_2(\mathbb{C})$ are

$$ad\ e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ ad\ f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},\ ad\ h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Now let $s, t \in \mathbb{C}$. Then

$$(s \cdot ad\ e)^3 = (t \cdot ad\ f)^3 = 0$$

so

$$exp(s \cdot ad\ e) = Id + s \cdot ad\ e + \tfrac{1}{2}(s \cdot ad\ e)^2 = \begin{pmatrix} 1 & -2s & -s^2 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix},$$

$$exp(t \cdot ad\ f) = Id + t \cdot ad\ f + \tfrac{1}{2}(t \cdot ad\ f)^2 = \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ -t^2 & 2t & 1 \end{pmatrix}.$$

The adjoint group G_{ad} generated by $exp(s \cdot ad\ e)$ and $exp(t \cdot ad\ f)$ is isomorphic to (a 3-dimensional representation of) $PSL_2(\mathbb{C})$.

Highest weight modules

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. A $\mathfrak{g}$–module V is called a *highest weight module* with *highest weight* $\lambda \in \mathfrak{h}^*$ if there exists $0 \neq v_\lambda \in V$ called a *highest weight vector* such that

$$\mathfrak{n}^+ \cdot v_\lambda = 0,$$

$$h \cdot v_\lambda = \lambda(h) v_\lambda$$

for $h \in \mathfrak{h}$ and

$$V = \mathcal{U}(\mathfrak{g})(v_\lambda).$$

Since $\mathfrak{n}_+$ annihilates v_λ and $\mathfrak{h}$ acts as scalar multiplication on v_λ, we have

$$V = \mathcal{U}(\mathfrak{n}_-)(v_\lambda).$$

If V^λ is a highest weight module with highest weight λ, then all weights of V^λ have the form

$$\lambda - \sum_{i=1}^{n} a_i \alpha_i$$

where α_i are the simple roots and $a_i \in \mathbb{Z}_{\geq 0}$.

Existence of highest weight modules

Let $\mathfrak{g}$ be a Lie algebra or Kac–Moody algebra. Then for all $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}$ has a highest weight module V with highest weight λ ([MP], Prop 2.3.1).

The highest weight vector $0 \neq v_\lambda \in V$ is unique up to nonzero scalar multiples kv_λ and the weight λ of v_λ is unique.

We have

$$V = \bigoplus_{\mu \in wts(V)} V_\mu, \quad dim(V_\mu) < \infty, \quad V_\lambda = \mathbb{C}v_\lambda.$$

Among all modules with highest weight $\lambda \in \mathfrak{h}^*$, there is a unique one that is irreducible as a $\mathfrak{g}$–module, that is, has no proper $\mathfrak{g}$–submodules except the trivial one ([K], Prop 9.3). We will often denote this by V^λ.

The module V^λ is integrable if and only if $\lambda \in P_+$, that is, λ is a dominant integral weight ([K], Lemma 10.1).

Summary: Highest weight modules

Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight. That is

$$\lambda \in P_+ = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0},\ i = 1, \ldots \ell\}.$$

Then there is a unique irreducible highest weight $\mathfrak{g}$–module V^λ.

Since λ is dominant, V^λ is integrable.

For Kac–Moody algebras $\mathfrak{g}$, the adjoint representation is integrable but is not a highest weight module.

For simple Kac–Moody algebras $\mathfrak{g}$, all $\mathfrak{g}$–modules $\{0\} \neq V$ are faithful, since $K = ker(\mathfrak{g} \to End(V))$ is an ideal, hence $K = \{0\}$.

FAITHFUL $\mathfrak{g}$–MODULES V^λ

The following lemma holds for both Lie algebras and Kac–Moody algebras $\mathfrak{g}$.

Lemma. *If V^λ is a faithful highest weight $\mathfrak{g}$–module with highest weight λ, then the lattice generated by the weights of V^λ contains the root lattice $Q = \mathbb{Z}\alpha_1 \oplus \ldots \mathbb{Z} \oplus \alpha_\ell$.*

Proof: Since V^λ is faithful, no e_i or f_i acts trivially on V^λ. So there exists $\mu \in wts(V^\lambda)$ such that e_i or f_i does not act trivially on V^λ_μ. We have

$$e_i : V^\lambda_\mu \to V^\lambda_{\mu+\alpha_i}$$

$$f_i : V^\lambda_\mu \to V^\lambda_{\mu-\alpha_i}$$

so $\mu + \alpha_i, \mu - \alpha_i \in wts(V^\lambda)$, and thus $\alpha_i \in wts(V^\lambda)$. It follows that the lattice generated by the weights of V^λ contains the root lattice. $\square$

Universal enveloping algebra $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$

Let $T(\mathfrak{g})$ denote the tensor algebra of $\mathfrak{g}$:

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$$

We may view elements of $T(\mathfrak{g})$ as formal noncommutative products, or 'power series' with $\mathfrak{g}$ as the single variable. We define

$$\mathcal{U}_{\mathbb{C}}(\mathfrak{g}) = T(\mathfrak{g})/I$$

where I is the two-sided ideal generated by elements of the form

$$a \otimes b - b \otimes a - [a, b] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \subset T(\mathfrak{g})$$

Then $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ is an associative algebra containing all possible polynomials in the e_i, f_i and $h \in \mathfrak{h}$ as well as all their products, subject to the natural relations in $\mathfrak{g}$.

$\mathbb{Z}$–FORM OF $\mathcal{U}_{\mathbb{C}}(\mathfrak{g})$

Let $\mathcal{U}_{\mathbb{C}} = \mathcal{U}_{\mathbb{C}}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Choose a lattice Λ between Q and P, that is $Q \leq \Lambda \leq P$, such that there are maps $i \mapsto \alpha_i$ and $i \mapsto \alpha_i^\vee$ from I to Λ and its $\mathbb{Z}$-dual $\Lambda^\vee$ where $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, $a_{ij} \in A$ and A is the generalized Cartan matrix of $\mathfrak{g}$.

$\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the $\mathbb{Z}$–subalgebra generated by the elements

$$\frac{e_i^m}{m!}, \frac{f_i^m}{m!}, \binom{h}{m} = \frac{h(h-1)\dots(h-m+1)}{m!}$$

for $i \in I, h \in \Lambda^\vee$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^+$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i^m}{m!}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^-$ be the $\mathbb{Z}$–subalgebra generated by $\dfrac{f_i^m}{m!}$ for $i \in I$ and $m \geq 0$,

$\mathcal{U}_{\mathbb{Z}}^0 \subseteq \mathcal{U}_{\mathbb{C}}(\mathfrak{h})$ be the $\mathbb{Z}$–subalgebra generated by $\binom{h}{m}$, for $h \in \Lambda^\vee$ and $m \geq 0$.

A $\mathbb{Z}$–FORM OF V^λ

Let $\mathbb{K}$ be a field. Let V^λ be the unique integrable highest weight $\mathfrak{g}$–module corresponding to the dominant integral weight λ.
We have

$$\mathcal{U}_{\mathbb{Z}}^{+} \cdot v_\lambda = \mathbb{Z} v_\lambda$$

since all elements of $\mathcal{U}_{\mathbb{Z}}^{+}$ except for 1 annihilate v_λ. Also

$$\mathcal{U}_{\mathbb{Z}}^{0} \cdot v_\lambda = \mathbb{Z} v_\lambda$$

$\mathcal{U}_{\mathbb{Z}}^{0}$ acts as scalar multiplication on v_λ by a $\mathbb{Z}$–valued scalar. Thus we have

$$\mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^{-} \cdot (\mathbb{Z} v_\lambda) = \mathcal{U}_{\mathbb{Z}}^{-} \cdot (v_\lambda).$$

We set

$$V_{\mathbb{Z}}^{\lambda} = \mathcal{U}_{\mathbb{Z}} \cdot v_\lambda = \mathcal{U}_{\mathbb{Z}}^{-} \cdot (v_\lambda)$$

Then $V_{\mathbb{Z}}^{\lambda}$ is a lattice in $V_{\mathbb{K}}^{\lambda} = \mathbb{K} \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^{\lambda}$ and a $\mathcal{U}_{\mathbb{Z}}$–module.

Simply connected Chevalley groups: overview

Chevalley constructed a $\mathbb{Z}$-form $\mathcal{U}_{\mathbb{Z}}$ of the universal enveloping algebra $\mathcal{U}$ of a complex simple Lie algebra $\mathfrak{g}$. This is a subring $\mathcal{U}_{\mathbb{Z}}$ of $\mathcal{U}_{\mathbb{C}}$ such that the canonical map

$$\mathcal{U}_{\mathbb{Z}} \otimes \mathbb{C} \longrightarrow \mathcal{U}_{\mathbb{C}}$$

is bijective. He then defined a $\mathbb{Z}$-form of $\mathfrak{g}$:

$$\mathfrak{g}_{\mathbb{Z}} = \mathfrak{g}_{\mathbb{C}} \cap \mathcal{U}_{\mathbb{Z}}.$$

For $\mathbb{K}$ an arbitrary field, we set

$$\mathcal{U}_{\mathbb{K}} = \mathcal{U}_{\mathbb{Z}} \otimes \mathbb{K}$$
$$\mathfrak{g}_{\mathbb{K}} = \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{K}.$$

Let V^{λ} be the irreducible highest weight module corresponding to a dominant integral weight λ.

A *simply connected Chevalley group* $G_{\mathbb{K}}$ is generated by elements of $Aut(V^{\lambda}_{\mathbb{K}})$, where $V^{\lambda}_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{Z}} V^{\lambda}_{\mathbb{Z}}$ for a $\mathbb{Z}$-form $V^{\lambda}_{\mathbb{Z}}$.

We will extend this construction to infinite dimensions.

(4) (Kac–Moody) Chevalley groups

Theorem([CL]) *Let $\mathfrak{g}$ be a symmetrizable Lie algebra or Kac–Moody algebra over $\mathbb{C}$. Let $\mathbb{K}$ be an arbitrary field. Let α_i, $i \in I$, be the simple roots and e_i, f_i the generators of $\mathfrak{g}$. Let $V^\lambda_{\mathbb{K}}$ be a $\mathbb{K}$–form of an integrable highest weight module V^λ for $\mathfrak{g}$, corresponding to dominant integral weight λ and defining representation $\rho : \mathfrak{g} \to End(V^\lambda_{\mathbb{K}})$. For $s, t \in \mathbb{K}$, let*

$$\chi_{\alpha_i}(s) = exp(\rho(se_i)),\ \chi_{-\alpha_i}(t) = exp(\rho(tf_i)).$$

Then

$$G^{V^\lambda}(\mathbb{K}) = \langle \chi_{\alpha_i}(s),\ \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{K} \rangle \leq Aut(V^\lambda_{\mathbb{K}})$$

is a simply connected (Kac–Moody) Chevalley group corresponding to $\mathfrak{g}$.

This construction is a natural generalization of the theory of elementary Chevalley groups over fields and can be extended to commutative rings (as in [Ch], [St]).

A similar construction for G^{V^λ} was used in [CG] to construct Kac–Moody groups over finite fields.

Simply connected Chevalley group corresponding to $\mathfrak{sl}_2$

We have

root lattice $Q = \mathbb{Z}\alpha$ and weight lattice $P = \mathbb{Z}\omega = \frac{1}{2}\mathbb{Z}\alpha$,

where α is the simple root. Thus $P/Q = \mathbb{Z}/2\mathbb{Z}$, hence the simply connected group is not isomorphic to the adjoint group.

Choose $\lambda = \omega$, where ω is the fundamental weight and let V^ω be the corresponding integrable highest weight module.

Let $\rho : \mathfrak{g} \to End(V^\omega)$ be the defining representation of V^ω. Let v^ω be a highest weight vector and let $V^\omega_{\mathbb{Z}}$ be the orbit of v^ω under $\mathcal{U}_{\mathbb{Z}}$

$$V^\omega_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \cdot v^\omega = \mathcal{U}^-_{\mathbb{Z}} \cdot v^\omega$$

where

$$\frac{(\rho(f))^n}{n!} \cdot v_\omega \in V_{\omega - n\alpha}, \; n \geq 0$$

where $V^\omega_{\omega-n\alpha}$ is the weight space of V^ω of weight $\omega - n\alpha$.

Simply connected Chevalley group corresponding to $\mathfrak{sl}_2$

Let $\mathbb{K}$ be an arbitrary field. The simply connected group $G^{V^\omega}(\mathbb{K}) < End(V^\omega)$ is the group $SL_2(\mathbb{K})$. That is,

$$\begin{aligned} G^{V^\omega}(\mathbb{K}) &= \langle \chi_\alpha(s), \chi_{-\alpha}(t) \mid s, t \in \mathbb{K} \rangle \\ &= \langle exp(s\rho(e)),\ exp(t\rho(f)) \mid s, t \in \mathbb{K} \rangle \\ &= \langle \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid s, t \in \mathbb{K} \rangle \\ &= SL_2(\mathbb{K}). \end{aligned}$$

For a $\mathbb{K}$-form $\mathfrak{g}_{\mathbb{K}} = \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{K}$ of $\mathfrak{g}_{\mathbb{C}}$, the adjoint group is

$$G^{V^\alpha}(\mathbb{K}) \cong PSL_2(\mathbb{K}).$$

Summary: Simply connected (Kac–Moody) Chevalley group

To construct a (Kac–Moody) Chevalley group, we will use the following data associated to a simple Lie algebra or Kac–Moody algebra $\mathfrak{g}$:

(i) a lattice Λ with $Q \leq \Lambda \leq P$, such that there are maps $i \mapsto \alpha_i$ and $i \mapsto \alpha_i^\vee$ from I to Λ and its $\mathbb{Z}$-dual $\Lambda^\vee$ where $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$, $a_{ij} \in A$.

(ii) a dominant integral weight $\lambda \in P_+$ and unique irreducible highest weight module V^λ,

(iii) $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$, a $\mathbb{Z}$-form of $\mathcal{U}_\mathbb{C}(\mathfrak{g})$, that is, the $\mathbb{Z}$-subalgebra generated by

$$\frac{e_i^m}{m!}, \frac{f_i^m}{m!}, i \in I, \binom{h}{m}, h \in \Lambda^\vee, m \geq 0,$$

(iv) $V_\mathbb{Z}^\lambda = \mathcal{U}_\mathbb{Z}(\mathfrak{g}) \cdot v_\lambda$, a $\mathbb{Z}$-form of V^λ,

(v) $V_\mathbb{K}^\lambda = \mathbb{K} \otimes_\mathbb{Z} V_\mathbb{Z}^\lambda$, for $\mathbb{K}$ an arbitrary field.

Dependence on choices

Our group constructions depend on

– A choice of lattice $Q \leq \Lambda \leq P$ between the root lattice Q and weight lattice P,

– A dominant integral weight λ, and an integrable highest weight module V^λ,

– A $\mathbb{Z}$-form $V^\lambda_{\mathbb{Z}}$.

The lattice Λ can be realized as the lattice of weights of a suitable representation V.

Conversely, the additive group generated by all the weights of a faithful representation V of $\mathfrak{g}$ forms a lattice $\Lambda = L_V$ between Q and P ([St], Lemma 27).

The simply connected group has desirable properties when we choose a highest weight module whose set of weights contains all the fundamental weights. For example:

– $L_V = Q$ if V is the adjoint representation, and

– $L_V = P$ if $V = V^{\omega_1 + \cdots + \omega_\ell}$, the highest weight module corresponding to the sum of the fundamental weights.

Dependence on choices: partial results

Finite dimensional Chevalley groups are independent of the choice of V^λ for $\lambda \in Q$ or $\lambda \in P$ and of the $\mathbb{Z}$-form $V^\lambda_{\mathbb{Z}}$ ([Hu], Ch 27).

Garland gave a representation theoretic construction of affine Kac–Moody groups as central extensions of loop groups, where each central extension corresponds to a unique cohomology class represented by a cocycle, known as the *Steinberg cocycle*.

He characterized the dependence on the choice of highest weight module V^λ for affine groups in terms of the Steinberg cocycle ([Ga1]).

For general Kac–Moody groups, the dependence on the choice of V^λ is not completely understood.

In [CW], the authors gave some preliminary results about the dependence of $G_{V^\lambda}(\mathbb{Z})$ on λ when $\mathfrak{g}$ is simply laced and hyperbolic.

For example, we conjectured that the discrepancy between the groups $E_{10}^{V^\lambda}(\mathbb{Z})$, as λ varies over the dominant integral weights, is contained in a finite abelian group of order at most $(\mathbb{Z}/2\mathbb{Z})^{10}$.

Some applications

Here are some choices of modules V for E_9, E_{10} and E_{11} that have physical relevance for the study of symmetries of supergravity and superstring theory.

Algebra	Highest weight module
$\mathfrak{e}_9(\mathbb{C})$	$V = V^{\omega_1}$ V integrable with high. wt. vector v^{ω_1} corresp. to fund. weight ω_1
$\mathfrak{e}_{10}(\mathbb{C})$	$V = V^{\omega_1+\cdots+\omega_{10}}$ V integrable with high. wt. vector $v^{\omega_1+\cdots+\omega_{10}}$
$\mathfrak{e}_{11}(\mathbb{C})$	$V = V^{\omega_{11}}$ V integrable with high. wt. vector $v^{\omega_{11}}$ corresp. to fund. weight ω_{11}

These choices give rise to group constructions which are useful in physical models. For example our generating set for E_{11} was used in [GW].