(Kac–Moody) Chevalley groups and Lie algebras with built–in structure constants

Lecture 3

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July 18, 2017

TOPICS

(1) Overview and introductory comments

(2) Lie algebras: finite and infinite dimensional

(3) Weights, representations and universal enveloping algebra

(4) (Kac–Moody) Chevalley groups

(5) (Kac–Moody) Chevalley groups over $\mathbb{Z}$, generators and defining relations for (Kac–Moody) Chevalley groups

(6) Structure constants for Kac–Moody algebras and Chevalley groups

Today we will answer the question of constructing (Kac–Moody) Chevalley groups over $\mathbb{Z}$ and of associating defining relations to (Kac–Moody) Chevalley groups.

Last time: (Kac–Moody) Chevalley groups

Let $\mathfrak{g}$ be a symmetrizable Lie algebra or Kac–Moody algebra over $\mathbb{C}$.

Let $\mathbb{K}$ be an arbitrary field. Let α_i, $i \in I$, be the simple roots and e_i, f_i the generators of $\mathfrak{g}$.

Let $V_{\mathbb{K}}^{\lambda}$ be a $\mathbb{K}$–form of an integrable highest weight module V^{λ} for $\mathfrak{g}$, corresponding to dominant integral weight λ and defining representation $\rho : \mathfrak{g} \to End(V_{\mathbb{K}}^{\lambda})$.

For $s, t \in \mathbb{K}$, let

$$\chi_{\alpha_i}(s) = exp(\rho(se_i)),\ \chi_{-\alpha_i}(t) = exp(\rho(tf_i)).$$

Then

$$G^{V^{\lambda}}(\mathbb{K}) = \langle \chi_{\alpha_i}(s),\ \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{K} \rangle \leq Aut(V_{\mathbb{K}}^{\lambda})$$

is a simply connected (Kac–Moody) Chevalley group corresponding to $\mathfrak{g}$.

Constructing a simply connected (Kac–Moody) Chevalley group

Recall that we chose a lattice $Q \leq \Lambda \leq P$ between the root lattice Q and weight lattice P, which can be realized as the lattice of weights of a suitable representation V.

The simply connected group has desirable properties when we choose a highest weight module whose set of weights contains all the fundamental weights.

If we choose $\Lambda = Q$ then G^V is the adjoint Chevalley group

If we choose $\Lambda = P$ then G^V is the simply connected Chevalley group

If $Q = P$, then a representation whose set of weights contains all the fundamental weights is $V = V^{\omega_1 + \cdots + \omega_\ell}$, the highest weight module corresponding to the sum of the fundamental weights.

The Chevalley group for $\mathfrak{sl}_3(\mathbb{C})$

Let $\mathfrak{sl}_3(\mathbb{C})$ denote the Lie algebra of 3×3 matrices of trace 0 over $\mathbb{C}$.

Let $\alpha_1,\ \alpha_2$ denote the simple roots. The basis of the standard 3 dimensional representation $\rho : \mathfrak{sl}_3(\mathbb{C}) \to End(V)$ of $\mathfrak{sl}_3(\mathbb{C})$ on $V = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ is:

$$x_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ h_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here we are using the Chevalley notation where $x_{\alpha_i} = e_i$, $x_{-\alpha_i} = f_i$.

The weights of ρ are:

$$\omega_1,\ \omega_2 - \omega_1,\ -\omega_2.$$

The representation ρ coincides with the highest weight module V^{ω_1} with highest weight ω_1.

Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ to Chevalley group $SL_3(\mathbb{C})$

Let $V = V^{\omega_1}$ be the highest weight module for $\mathfrak{sl}_3(\mathbb{C})$. Let $\rho : \mathfrak{sl}_3(\mathbb{C}) \to End(V^{\omega_1})$ be the defining representation. Let $s, t \in \mathbb{C}$. In $Aut(V^{\omega_1})$, as before, we set

$$\chi_{\alpha_i}(s) = exp(\rho(sx_{\alpha_i})), \quad \chi_{-\alpha_i}(t) = exp(\rho(tx_{-\alpha_i}))$$

But $x^2_{\alpha_1} = x^2_{-\alpha_1} = 0$ and $x^2_{\alpha_2} = x^2_{-\alpha_2} = 0$ thus

$$\chi_{\alpha_1}(s) = Id + \rho(sx_{\alpha_1}) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \chi_{-\alpha_1}(t) = Id + \rho(tx_{-\alpha_1}) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\chi_{\alpha_2}(s) = Id + \rho(sx_{\alpha_2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad \chi_{-\alpha_2}(t) = Id + \rho(tx_{-\alpha_2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

The lattice generated by the weights of V^{ω_1} is the weight lattice P. Thus the Chevalley group $G^{V^{\omega_1}}$ is simply connected.

Simply connected Chevalley group corresponding to $\mathfrak{sl}_3(\mathbb{C})$

The simply connected Chevalley group is the group $G^{V^{\omega_1}} \leq Aut(V^{\omega_1})$, generated by the automorphisms $\chi_{\pm\alpha_i}$:

$$\begin{aligned} G^{V^{\omega_1}}(\mathbb{C}) &= \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid i = 1, 2,\ s, t \in \mathbb{C} \rangle \\ &= \langle exp(\rho(s x_{\alpha_i})),\ exp(\rho(t x_{-\alpha_i})) \mid i = 1, 2,\ s, t \in \mathbb{C} \rangle \\ &= \left\langle \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \mid s, t \in \mathbb{C} \right\rangle \end{aligned}$$

This is the simple Lie group $SL_3(\mathbb{C})$.

Arithmetic subgroup $G^V(\mathbb{Z})$

The arithmetic subgroup $SL_n(\mathbb{Z})$ of $SL_n(\mathbb{C})$ is obtained by taking $\mathbb{Z}$–entries in the matrix representation of $SL_n(\mathbb{C})$.

This corresponds to taking '$\mathbb{Z}$–points'

$$G^V(\mathbb{Z}) = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{Z},\ i \in I \rangle$$

of the Chevalley group $G^V(\mathbb{C})$.

For $G^V(\mathbb{C}) = SL_2(\mathbb{C})$, this is the subgroup $G^V(\mathbb{Z}) = SL_2(\mathbb{Z})$ generated by the matrices

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

for $s, t \in \mathbb{Z}$.

This is well known, but does not generalize to exceptional groups or to Kac–Moody groups.

A crucial fact for generalizing this construction is the following.

The subgroup $SL_2(\mathbb{Z})$ of $SL_2(\mathbb{C})$ is also the stabilizer of a $\mathbb{Z}$–form $V_{\mathbb{Z}}$ of the standard representation $V_{\mathbb{C}}$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Hidden structure

Proposition The subgroup $SL_2(\mathbb{Z})$ of $SL_2(\mathbb{C})$ is the stabilizer of a $\mathbb{Z}$-form $V_\mathbb{Z}$ of the standard representation $V_\mathbb{C}$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Proof: Take $V_\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$ and $V_\mathbb{C} = \mathbb{C} \oplus \mathbb{C}$. Then $SL_2(\mathbb{Q})$ acts on $V_\mathbb{C}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Suppose now that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stabilizes $V_\mathbb{Z}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z}, \text{ that is } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $ad - bc = 1, x, y \in \mathbb{Z}$ and $u = ax + by \in \mathbb{Z}, v = cx + dy \in \mathbb{Z}$.

Take $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $u = ax + by$ implies $b \in \mathbb{Z}$ and $v = cx + dy$ implies $d \in \mathbb{Z}$.

Take $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then $u = ax + by$ implies $a \in \mathbb{Z}$ and $v = cx + dy$ implies $c \in \mathbb{Z}$.

Thus if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z}$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. $\square$

‘Arithmetic subgroup’ of a (Kac–Moody) Chevalley group

For finite dimensional Chevalley groups, Chevalley defined the arithmetic subgroup $G^{V^\lambda}(\mathbb{Z})$ as follows:

$$G^{V^\lambda}(\mathbb{Z}) = \{g \in G^{V^\lambda}(\mathbb{C}) \mid g(V_\mathbb{Z}) = V_\mathbb{Z}\} \leq Aut(V_\mathbb{Z}).$$

This is the subgroup of $G^{V^\lambda}(\mathbb{C})$ preserving the lattice $V^\lambda_\mathbb{Z}$ in the representation space V^λ.
How does this compare with the ‘group of $\mathbb{Z}$–points’

$$G_\mathbb{Z} = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{Z},\ i \in I \rangle$$

of

$$G_\mathbb{C} = \langle \chi_{\alpha_i}(s), \chi_{-\alpha_i}(t) \mid s, t \in \mathbb{C},\ i \in I \rangle?$$

When $\mathfrak{g}$ is finite dimensional, it is straightforward to prove that

$$G^{V^\lambda}(\mathbb{Z}) \cong G_\mathbb{Z}.$$

When $\mathfrak{g}$ is an infinite dimensional Kac–Moody algebra, this is a difficult and substantial theorem, proven by C-Liu.

SUBTLE ISSUES

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ is written in terms of the generators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_{\pm\alpha}(t_1)\chi_{\pm\alpha}(t_2)\dots\chi_{\pm\alpha}(t_k)$$

then it is not necessarily the case that the scalars t_i are all integers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \chi_\alpha(\frac{1}{2})h_\alpha(\frac{1}{2})\chi_{-\alpha}(\frac{1}{2}) = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix}.$$

However, since $g \in SL_2(\mathbb{Z})$ and the $\chi_{\pm\alpha}(t)$ generate $SL_2(\mathbb{Z})$ for $t \in \mathbb{Z}$, there exist integers $s_1, \dots, s_n$ such that

$$g = \chi_{\pm\alpha}(s_1)\chi_{\pm\alpha}(s_2)\dots\chi_{\pm\alpha}(s_n).$$

THE CHEVALLEY GROUP $E_7(\mathbb{Z})$

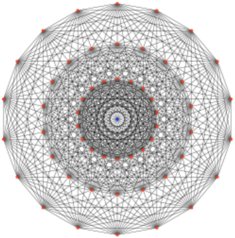
Hull and Townsend, following Cremmer and Julia, discovered the following form of $E_7(\mathbb{Z})$:

$$E_{7(+7)}(\mathbb{Z}) = E_{7(+7)}(\mathbb{R}) \cap Sp(56, \mathbb{Z})$$

in the framework of type II superstring theory. Soulé gave a rigorous mathematical proof that the $E_{7(+7)}(\mathbb{Z})$ of Hull and Townsend coincides with the Chevalley $\mathbb{Z}$–form of $G = E_7$ given by

$$E_7(\mathbb{Z}) = \{g \in E_7(\mathbb{C}) \mid g(V_{\mathbb{Z}}) = V_{\mathbb{Z}}\} \leq Aut(V_{\mathbb{Z}}).$$

Here $V_{\mathbb{Z}}$ is the stabilizer of the standard lattice in the unique 56–dimensional fundamental representation of E_7.



The E_7 Gosset polytope 3_{21}

GENERATING SETS

Theorem ([CL]) Let R be a commutative ring with 1. Let λ be a dominant integral weight and let V^λ be the corresponding integrable highest weight module with simply connected Kac–Moody Chevalley group

$$G^{V^\lambda}(R) = \langle exp(\rho(se_i)),\ exp(\rho(tf_i)) \mid s, t \in R \rangle$$

Let $s, t \in R$, $u \in R^\times$ and set

$$\chi_{\alpha_i}(s) = exp(\rho(se_i)), \quad \chi_{-\alpha_i}(t) = exp(\rho(tf_i)),$$

$$\widetilde{w}_{\alpha_i}(u) = \chi_{\alpha_i}(u)\chi_{-\alpha_i}(-u^{-1})\chi_{\alpha_i}(u), \quad h_{\alpha_i}(u) = \widetilde{w}_{\alpha_i}(u)\widetilde{w}_{\alpha_i}(1)^{-1}.$$

Then $G^{V^\lambda}(R)$ has the following generating sets:
(1) $\chi_{\alpha_i}(s)$ and $\chi_{-\alpha_i}(t)$,
and
(2) $\chi_{\alpha_i}(s)$ and $\widetilde{w}_{\alpha_i}(1) = \chi_{\alpha_i}(1)\chi_{-\alpha_i}(-1)\chi_{\alpha_i}(1)$.

Generating sets for $SL_2(\mathbb{Z})$

The simply connected Chevalley group $SL_2(\mathbb{Z})$ has the following generating sets

(1) $\chi_\alpha(1)$ and $\chi_{-\alpha}(1)$, corresponding to matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(2) $\chi_\alpha(1)$ and $\widetilde{w}_\alpha(1) = \chi_\alpha(1)\chi_{-\alpha}(-1)\chi_\alpha(1)$, corresponding to matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $s \in \mathbb{Z}$.

GENERATORS AND RELATIONS FOR CHEVALLEY GROUPS

Steinberg gave a defining presentation for finite dimensional Chevalley groups over commutative rings R, using the generating sets that we have described.

Tits gave generators and relations for Kac–Moody groups, generalizing the Steinberg presentation.

In the finite dimensional case, there is a Chevalley type commutation relation of the form

$$(\chi_\alpha(u), \chi_\beta(v)) = \prod_{m,n} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta} u^m v^n)$$

between every pair of elements χ_α, χ_β.

Here $u, v \in R$, $C_{mn\alpha\beta}$ are integers and the χ_α are viewed as formal symbols in

$$U_\alpha = \{\chi_\alpha(u) \mid \alpha \in \Delta,\ u \in R\} \cong (R, +).$$

However, in the infinite dimensional case, Tits' presentation of Kac–Moody groups has infinitely many Chevalley commutation relations.

Determining Tits' Kac–Moody group presentation

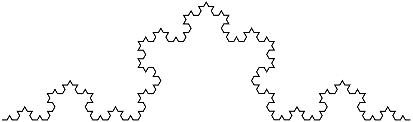
In the infinite dimensional Kac–Moody case, Tits determined that whenever a pair of real roots is 'prenilpotent', then there is a Chevalley commutation relation necessary for defining the Kac–Moody group. In order to make Tits' presentation complete, we need to:

Explicitly describe the infinite set of prenilpotent pairs of roots.

This usually requires us to:

Explicitly describe the infinite set of positive real roots.

There is no guarantee that either of these tasks can be carried out in practice.



Prenilpotent pairs

Let (α, β) be a pair of real roots and let W denote the Weyl group. Then (α, β) is called a *prenilpotent pair*, if there exist $w,\ w' \in W$ such that

$$w\alpha,\ w\beta \in \Delta_+^{re} \text{ and } w'\alpha,\ w'\beta \in \Delta_-^{re}.$$

A pair of roots $\{\alpha, \beta\}$ is prenilpotent if and only if $\alpha \neq -\beta$ and

$$(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Delta_+^{re}$$

is a finite set. For every prenilpotent pair of roots $\{\alpha, \beta\}$, Tits defined the Chevalley commutation relation

$$(\chi_\alpha(u), \chi_\beta(v)) = \prod_{m\alpha+n\beta \in (\mathbb{Z}_{>0}\alpha+\mathbb{Z}_{>0}\beta)\cap\Delta_+^{re}} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta}u^m v^n)$$

where $u, v \in R$ and $C_{mn\alpha\beta}$ are integers.

If $\mathfrak{g}$ is finite dimensional, every pairs (α, β) of roots is prenilpotent.

THE TITS–STEINBERG PRESENTATION

Let R be a commutative ring with 1. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra or symmetrizable Kac–Moody algebra.
We may associate to $\mathfrak{g}$ a group G over R, generated by the set of symbols $\{\chi_\alpha(u) \mid \alpha \in \Delta^{re}, u \in R\}$ satisfying relations (R1)–(R7) below.
Let $i, j \in I$, $u, v \in R$ and $\alpha, \beta \in \Delta^{re}$.

(R1) $\chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v)$;

(R2) For each prenilpotent pair (α, β),

$$(\chi_\alpha(u), \chi_\beta(v)) = \prod_{\substack{m,n>0 \\ m\alpha+n\beta \in \mathbb{Z}\alpha \oplus \mathbb{Z}\beta}} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta} u^m v^n)$$

where $C_{mn\alpha\beta}$ are integers.

(R3) $\widetilde{w}_i \chi_\alpha(u) \widetilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i} u)$,

(R4) $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{\langle \alpha, \alpha_i^\vee \rangle})$ for $u \in R^*$,

(R5) $\widetilde{w}_i h_j(u) \widetilde{w}_i^{-1} = h_j(u) h_i(u^{-a_{ji}})$,

(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in R^*$, and

(R7) $(h_i(u), h_j(v)) = 1$ for $u, v \in R^*$.

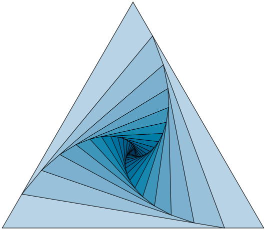
THE DIFFICULTY OF FINDING PRENILPOTENT PAIRS

For $\mathfrak{g} = \mathfrak{e}_{10}$, work of Allcock has shown that the problem of determining all prenilpotent pairs is tractable but impractical.

Namely, Allcock showed that for the number of W–orbits of prenilpotent pairs of real roots having inner product equal to k grows at least as fast as $(const)k^7$ as $k \to \infty$.

For each such orbit, we then have to enumerate the prenilpotent pairs of real roots.

We need a different approach.



Simplifying Tits' presentation

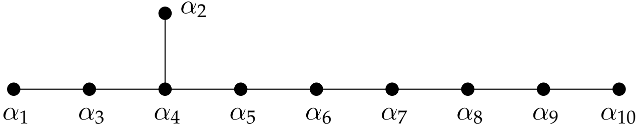
Tits' presentation is very redundant and can be reduced significantly.

In joint work with D. Allcock, we obtained a simplification of Tits' presentation for Kac–Moody algebras that are simply laced and hyperbolic, giving a *finite* presentation for these groups over $\mathbb{Z}$.

A Dynkin diagram is *simply laced* if it consists only of single bonds between nodes.

This class includes the group $E_{10}(\mathbb{Z})$, conjectured to be a discrete symmetry group of Type II superstring theory.

Our result shows that in a simply laced hyperbolic Kac–Moody group, the Chevalley commutation relations arising from *simple roots* imply the Chevalley commutation relations arising from all real roots.



A FINITE PRESENTATION FOR $G(\mathbb{Z})$

Let $\mathfrak{g}$ be a simply laced, symmetrizable and hyperbolic Kac–Moody algebra.

Tits' presentation can be reduced to the following *finite* presentation for $G(\mathbb{Z})$:

Over $R = \mathbb{Z}$, the generators $X_i(u)$ are obtained from $X_i = X_i(1)$ via $X_i(u) = X_i^u$.

Any i	Each $i = \{1, \dots, \ell\}$	$i \neq j$ not adjacent	$i \neq j$ adjacent
$S_i^4 = 1$	$[S_i^2, X_i] = 1$	$S_iS_j = S_jS_i$	$S_iS_jS_i = S_jS_iS_j$ $S_i^2S_jS_i^{-2} = S_j^{-1}$
	$S_i = X_iS_iX_iS_i^{-1}X_i$	$[S_i, X_j] = 1$	$X_iS_jS_i = S_jS_iX_j$ $S_i^2X_jS_i^{-2} = X_j^{-1}$
		$[X_i, X_j] = 1$	$[X_i, X_j] = S_iX_jS_i^{-1}$ $[X_i, S_iX_jS_i^{-1}] = 1$

Table: The defining relations for $G(\mathbb{Z})$, G simply laced and hyperbolic

Comparing Kac–Moody groups

We have been considering the following group constructions:

– Tits' presentation, which defines a Kac–Moody group,
– Our Kac–Moody Chevalley group $G^V(\mathbb{C})$, for some choice of V

We conjecture that over $\mathbb{C}$, Tits' group given by generators and relations coincides with our Kac–Moody Chevalley group $G^V(\mathbb{C})$ for some choice of highest weight module V.

This involves the functorial properties of Tits' Kac–Moody group, which are not completely understood.

In the finite dimensional case, this isomorphism is well known.

THE FEINGOLD–FRENKEL HYPERBOLIC KAC–MOODY ALGEBRA AE_3

Consider the hyperbolic Kac–Moody algebra with generalized Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

It has been conjectured for some time that this hyperbolic algebra may be an algebra of internal symmetries of Einstein's gravitational equations.
The Weyl group W is the $(3, 2, \infty)$-triangle group:

$$W = \langle w_1, w_2, w_3 \mid w_i^2 = 1,\ (w_2w_3)^3 = 1, (w_1w_3)^2 = 1\rangle,$$

which is isomorphic to $PGL_2(\mathbb{Z})$.The Dynkin diagram

is not simply laced and hence does not satisfy the conditions to give rise to our finite presentation.
In recent work with Scott Murray, we give a recursive method for determining the infinite set of prenilpotent pairs of real roots in AE_3.