(Kac–Moody) Chevalley groups and Lie algebras with built–in structure constants

Lecture 4

Lisa Carbone, Rutgers University
lisa.carbone@rutgers.edu

July 25, 2017

TOPICS

(1) Overview and introductory comments

(2) Lie algebras: finite and infinite dimensional

(3) Weights, representations and universal enveloping algebra

(4) (Kac–Moody) Chevalley groups

(5) Generators and relations and (Kac–Moody) Groups over $\mathbb{Z}$

(6) Structure constants for Kac–Moody algebras and Chevalley groups

Today we will answer the question of how to determine the structure constants for Kac–Moody algebras. We will also discuss some new results in the finite dimensional case.

Structure constants: definition

Let $\mathfrak{g}$ be a Lie algebra or Kac–Moody algebra. Viewing $\mathfrak{g}$ as a vector space over $\mathbb{C}$ with a bilinear operation

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (x, y) &\mapsto [x, y] \end{aligned}$$

the structure constants of $\mathfrak{g}$ are the constants that occur in the evaluation of the Lie bracket in terms of a choice of basis for $\mathfrak{g}$. If $\mathfrak{g}$ has basis $\{x_i\}_{i=1,\ldots}$ and Lie bracket $[\cdot, \cdot]$ defined by

$$[x_i, x_j] = \sum_k n_{ijk} x_k.$$

then the elements $n_{ijk} \in \mathbb{C}$ are the *structure constants* of $\mathfrak{g}$ and they depend on the choice of basis for $\mathfrak{g}$.

Structure constants: some basic properties

It turns out that we don't have to use all basis vectors x_i in the expression $[x_i, x_j] = \sum_k n_{ijk} x_k$.

For any pair of roots $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$

It follows that in the expression

$$[x_\alpha, x_\beta] = \sum_k n_{\alpha,\beta,k}\, x_k,$$

only one summand $n_{\alpha,\beta} x_{\alpha+\beta}$ occurs. We write

$$[x_\alpha, x_\beta] = n_{\alpha,\beta} x_{\alpha+\beta}.$$

Structure constants: some basic properties

A system of structure constants over $\mathbb{C}$, satisfying the obvious relations given by skew symmetry and the Jacobi identity, can be used to fully determine the multiplication table of a finite dimensional Lie algebra.

For any pair of roots $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Delta$ we have

$$[x_\alpha, x_\beta] = n_{\alpha,\beta} x_{\alpha+\beta}.$$

If $\alpha + \beta$ is not a root, then we set $n_{\alpha,\beta} = 0$.

For all $\alpha \in \Delta$, $[x_\alpha, x_\alpha] = 0$, thus we set $n_{\alpha,\alpha} = 0$.

Since $[x_\alpha, x_\beta] = -[x_\beta, x_\alpha]$, we have $n_{\alpha,\beta} = -n_{\beta,\alpha}$ for all roots α, β.

The Jacobi identity gives rise to relations between the structure constants of the following type:

$$n_{\alpha+\beta,\gamma} n_{\alpha,\beta} = -n_{\beta,\gamma} n_{\beta+\gamma,\alpha}.$$

Two finite dimensional Lie algebras are isomorphic if and only if they have bases which give rise to the same systems of structure constants.

Signs in the Lie bracket of $\mathfrak{sl}_3(\mathbb{C})$

Let $\mathfrak{sl}_3(\mathbb{C})$ denote the Lie algebra of 3×3 matrices of trace 0 over $\mathbb{C}$. The simple roots are $\alpha_1,\ \alpha_2$. We choose a basis

$$x_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ h_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$x_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Signs in the Lie bracket of $\mathfrak{sl}_3(\mathbb{C})$

This is a *Chevalley basis* for $\mathfrak{sl}_3(\mathbb{C})$. Relative to this basis, all structure constants are integers and we have the Lie bracket:

$$[h_{\alpha_i}, h_{\alpha_j}] = 0$$

$$[h_{\alpha_i}, x_{\alpha_j}] = \langle h_{\alpha_i}, x_{\alpha_j} \rangle x_{\alpha_j}$$

$$[x_{\alpha_i}, x_{\alpha_j}] = \begin{cases} n_{\alpha_i,\alpha_j} x_{\alpha_i+\alpha_j}, & \text{if } \alpha_i + \alpha_j \in \Delta, \\ -h_{\alpha_i}, & \text{if } \alpha_i = -\alpha_j, \\ 0, & \text{otherwise.} \end{cases}$$

Here $n_{\alpha_i,\alpha_j} \in \{\pm 1\}$.

Structure constants in $\mathfrak{sl}_3(\mathbb{C})$ and $SL_3(\mathbb{C})$

But $\alpha_1 + \alpha_2$ is a root in $\mathfrak{sl}_3(\mathbb{C})$ and so we have

$$[x_{\alpha_1}, x_{\alpha_2}] = n_{\alpha_1,\alpha_2} x_{\alpha_1+\alpha_2}$$

where $x_{\alpha_1+\alpha_2}$ is a root vector for $\alpha_1 + \alpha_2$. For $\mathfrak{sl}_3(\mathbb{C})$, we have $n_{\alpha_1,\alpha_2} \in \{\pm 1\}$. Thus the Lie algebra commutator

$$[x_{\alpha_1}, x_{\alpha_2}] = \pm x_{\alpha_1+\alpha_2}$$

is defined only up to a sign. This sign ambiguity also appears in the group commutator in the simply connected Chevalley group $SL_3(\mathbb{C})$:

$$(\chi_{\alpha_1}(s), \chi_{\alpha_2}(t)) = \chi_{\alpha_1+\alpha_2}(\pm st)$$

It is known that there exists a choice of signs making the commutation relations consistent with the Lie bracket across $\mathfrak{sl}_3(\mathbb{C})$ (and hence $SL_3(\mathbb{C})$).

Sign ambiguity and central extension

Chevalley showed that the structure constants of any finite dimensional semisimple Lie algebra are determined up to ± 1.

Tits showed that the sign cannot be disregarded due to the existence of a canonical central extension of the root lattice by a cyclic group of order 2.

Choosing signs for the Lie bracket corresponds to choosing a section of the central extension.

This also holds in the infinite dimensional Kac–Moody case.

Determining a consistent system of signs of structure constants in Lie algebras and Kac–Moody algebras is a persistent problem in Lie theory, computational algebra, computational number theory and their applications.

Many mathematicians are left consulting pages of tables like this one

Vavilov (2004): **Structure constants for E_7**

	1000000	0100000	0010000	0001000	0000100	0000010	0000001	1010000	0101000	0011000	0001100	0000110	0000011	1011000	0111000	0101100	0011100	0001110	0000111	1111000	1011100	0111100	0101110	0011110	0001111	1111100	1011110	0112100	0111110	0101111	0011111	1112100
1000000	0	0	+	0	0	0	0	0	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+	0	0	0	+	+	0	+	0
0100000	0	0	0	+	0	0	0	0	0	+	+	0	0	+	0	0	+	+	0	0	+	0	0	+	+	0	+	0	0	0	+	0
0010000	−	0	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+
0001000	0	−	−	0	+	0	0	−	0	0	0	+	0	0	0	0	0	0	+	0	0	+	0	0	0	+	0	0	+	0	0	0
0000100	0	0	0	−	0	+	0	0	−	−	0	0	+	−	−	0	0	0	0	−	0	0	0	0	0	0	0	0	0	0	0	0
0000010	0	0	0	0	−	0	+	0	0	0	−	0	0	0	0	−	−	0	0	0	−	−	0	0	0	−	0	−	0	0	0	−
0000001	0	0	0	0	0	−	0	0	0	0	0	−	0	0	0	0	0	−	0	0	0	0	−	−	0	0	−	0	−	0	0	0
1010000	0	0	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+	0	0	0	0	+	0	+	0	0	−	0	+	0	0
0101000	0	0	−	0	+	0	0	−	0	0	0	+	0	0	0	0	−	0	+	0	−	0	0	−	0	0	−	0	0	0	−	0
0011000	−	−	0	0	+	0	0	0	0	0	0	+	0	0	0	−	0	0	+	0	0	0	−	0	0	+	0	0	0	−	0	0
0001100	0	−	−	0	0	+	0	−	0	0	0	0	+	0	−	0	0	0	0	−	0	0	0	0	0	0	0	0	−	0	0	0
0000110	0	0	0	−	0	0	+	0	−	−	0	0	0	−	−	0	0	0	0	−	0	0	0	0	0	0	0	−	0	0	0	−
0000011	0	0	0	0	−	0	0	0	0	0	−	0	0	0	0	−	−	0	0	0	−	−	0	0	0	−	0	−	0	0	0	−
1011000	0	−	0	0	+	0	0	0	0	0	0	+	0	0	0	−	0	0	+	0	0	−	−	0	0	0	0	0	−	−	0	0
0111000	−	0	0	0	+	0	0	0	0	0	+	+	0	0	0	0	0	+	+	0	−	0	0	0	+	0	−	0	0	0	0	0
0101100	0	0	−	0	0	+	0	−	0	+	0	0	+	+	0	0	0	0	0	0	0	0	0	+	0	0	+	0	0	0	+	0
0011100	−	−	0	0	0	+	0	0	+	0	0	0	+	0	0	0	0	0	0	−	0	0	+	0	0	0	0	0	0	+	0	0
0001110	0	−	−	0	0	0	+	−	0	0	0	0	0	0	−	0	0	0	0	−	0	+	0	0	0	+	0	0	0	0	0	0
0000111	0	0	0	−	0	0	0	0	−	−	0	0	0	−	−	0	0	0	0	−	0	0	0	0	0	0	0	−	0	0	0	−
1111000	0	0	0	0	+	0	0	0	0	0	+	+	0	0	0	0	+	+	+	0	0	0	0	+	+	0	0	0	0	0	+	0
1011100	0	−	0	0	0	+	0	0	+	0	0	0	+	0	+	0	0	0	0	0	0	0	+	0	0	0	0	0	+	+	0	0
0111100	−	0	0	−	0	+	0	0	0	0	0	0	+	+	0	0	0	−	0	0	0	0	0	0	−	0	+	0	0	0	0	0
0101110	0	0	−	0	0	0	+	−	0	+	0	0	0	+	0	0	−	0	0	0	−	0	0	0	0	0	0	0	0	0	−	0
0011110	−	−	0	0	0	0	+	0	+	0	0	0	0	0	0	−	0	0	0	−	0	0	0	0	0	+	0	0	0	−	0	+
0001111	0	−	−	0	0	0	0	−	0	0	0	0	0	0	−	0	0	0	0	−	0	+	0	0	0	+	0	0	−	0	0	0
1111100	0	0	0	−	0	+	0	0	0	−	0	0	+	0	0	0	0	−	0	0	0	0	0	−	−	0	0	0	0	0	−	0
1011110	0	−	0	0	0	0	+	0	+	0	0	0	0	0	+	−	0	0	0	0	0	−	0	0	0	0	0	−	0	−	0	0
0112100	−	0	0	0	0	+	0	+	0	0	0	+	+	0	0	0	0	0	+	0	0	0	0	0	0	0	+	0	0	0	0	0
0111110	−	0	0	−	0	0	+	0	0	0	+	0	0	+	0	0	0	0	0	0	−	0	0	0	+	0	0	0	0	0	0	+
0101111	0	0	−	0	0	0	0	−	0	+	0	0	0	+	0	0	−	0	0	0	−	0	0	+	0	0	+	0	0	0	0	0
0011111	−	−	0	0	0	0	0	0	+	0	0	0	0	0	0	−	0	0	0	−	0	0	+	0	0	+	0	0	0	0	0	+
1112100	0	0	−	0	0	+	0	0	0	0	0	+	+	0	0	0	0	0	+	0	0	0	0	−	0	0	0	0	−	0	−	0

Roots occur in strings

Our method for determining structure constants follows Chevalley's use of root strings.

If $\alpha, \beta \in \Delta$, then the α root string through β

$$-p_{\alpha,\beta}\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q_{\alpha\beta}\alpha + \beta$$

is an unbroken string of roots and

$$\langle \beta, \alpha \rangle = p_{\alpha,\beta} - q_{\alpha\beta}.$$

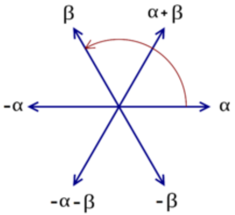
For example, the roots of A_2 are

$$\alpha,\ \beta,\ \alpha + \beta,\ -\alpha,\ -\beta,\ -\alpha - \beta.$$

The α root string through β is

$$\beta,\ \alpha + \beta.$$

That is, $\beta - \alpha$ and $\beta + 2\alpha$ are not roots.



Reduction to rank 2 root subsystems

To determine structure constants, we will reduce to rank 2 root subsystems.

Let α and β be (real) roots and let $\Delta(\alpha, \beta)$ denote the rank 2 root subsystem generated by α and β. That is,

$$\Delta(\alpha, \beta) = W_{\{\alpha,\beta\}}\{\alpha, \beta\},$$

where $W_{\{\alpha,\beta\}}$ is the Weyl group generated by reflections w_α and w_β.

The structure constant $n_{\alpha,\beta}$ in a semisimple Lie algebra or Kac–Moody algebra may be computed in the rank 2 root subsystem $\Delta(\alpha, \beta)$ generated by α and β.

Reduction to rank 2 subsystems

The possibilities for rank 2 subsystems $\Delta(\alpha,\beta) = W_{\{\alpha,\beta\}}\{\alpha,\beta\}$ are obtained by looking at the classification of rank 2 (generalized) Cartan matrices:

Rank 2 finite type:

$$A_1 \times A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\ A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},\ B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},\ G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Rank 2 Kac–Moody type:

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},\ A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix},$$

$$H(m) = \begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}_{m\in\mathbb{Z}_{\geq 3}},\ H(a,b) = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}_{ab>4}$$

Chevalley's theorem

Chevalley's theorem on structure constants is the following.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Suppose that α, β, $\alpha + \beta \in \Delta$. Let

$$-p_{\alpha,\beta}\alpha + \beta, \ldots, -\alpha + \beta, \beta, \beta + \alpha, \ldots \beta + q_{\alpha\beta}\alpha$$

be the α root string through β. Then there is a basis for $\mathfrak{g}$ with respect to which

(1) $n_{-\alpha,-\beta} = n_{\alpha,\beta}$.

(2) $n_{\alpha,\beta} = \pm(p_{\alpha,\beta} + 1)$.

We write $n_{\alpha,\beta} = s_{\alpha,\beta}(p_{\alpha,\beta} + 1)$ where $s_{\alpha,\beta} \in \{\pm 1\}$.

Chevalley's theorem shows that there is a basis for $\mathfrak{g}$ with respect to which the structure constants are integers.

Morita showed that (1) and (2) hold for Kac–Moody algebras when $\alpha, \beta, \alpha + \beta \in \Delta^{re}$.

How to determine the structure constants?

Since $n_{\alpha,\beta} = s_{\alpha,\beta}(p_{\alpha,\beta} + 1)$ where $s_{\alpha,\beta} \in \{\pm 1\}$ and $p_{\alpha,\beta}$ is determined by the α root string through β, to find $n_{\alpha,\beta}$ we have to find:

· all pairs of (real) roots whose sum is a (real) root

· all root strings and all $p_{\alpha,\beta}$ for $\alpha, \beta, \alpha + \beta \in \Delta^{re}$

· a system of signs $\{s_{\alpha,\beta} \mid \alpha, \beta \in \Delta^{re}\}$ such that $n_{\alpha,\beta}$ satisfies relations implied by skew symmetry and the Jacobi identity.

Structure constants in finite dimensions

The most familiar algorithm for computing structure constants for finite dimensional simple Lie algebras is given by Carter. This algorithm has been implemented in GAP by De Graaf and in Magma by Cohen, Haller and Murray.

There are also algorithms by Gilkey and Seitz and Vavilov for the exceptional groups E_n, $n = 6, 7, 8$. Rylands gave an analogue of these algorithms for F_4 and G_2 and her formulas were implemented in Magma.

For simply laced Lie algebras, Kac and Frenkel and Kac gave an elegant (implicit) method for determining structure constants, developed further by Frenkel, Lepowsky and Meurman.

Tits observed that one could compute the structure constants in the extended Weyl group $\widetilde{W}$. This observation has been expanded to an algorithm by Casselman.

Structure constants: Finite dimensional case

We have $n_{\alpha,\beta} = s_{\alpha,\beta}(p_{\alpha,\beta} + 1)$, where $p_{\alpha,\beta}$ is determined by the α root string through β.

Kac, Frenkel, Lepowsky and Meurman, Frenkel and Kac implicitly defined structure constants in terms of bilinear forms $s_0(\alpha, \beta)$ in the simply laced case (A_n, D_n, E_n), where

$$s_{\alpha,\beta} = (-1)^{s_0(\alpha,\beta)}$$

and $s_0(\alpha, \beta)$ satisfies natural conditions arising from skew-symmetry and the Jacobi identity.

Lepowsky and Primc suggested that bilinear forms for the structure constants for B_n, G_2 could be obtained from their methods.

In joint work with Coulson, Kanade, McRae and Murray, we give explicit bilinear forms $s_0(\alpha, \beta)$ in many cases.

STRUCTURE CONSTANTS FOR (KAC–MOODY) CHEVALLEY GROUPS

Let $\mathfrak{g}$ be a symmetrizable simple Lie algebra or Kac–Moody algebra. Let G^V be the Kac–Moody Chevalley group corresponding to an integrable highest weight module V. Let α, β be real roots whose sum $\alpha+\beta$ is a real root. Let $s, t \in \mathbb{C}$. Then

$$(\chi_\alpha(s), \chi_\beta(t)) = \prod_{(m\alpha+n\beta)\cap\Delta_+^{re}} \chi_{m\alpha+n\beta}(c_{\alpha,\beta,s,t}t^m s^n).$$

Theorem ([C]) If $\mathfrak{g}$ is simply laced, then

$$\{(m\alpha+n\beta)\cap\Delta_+^{re}\} \subseteq \{\alpha+\beta\}$$

$$(\chi_\alpha(s), \chi_\beta(t)) = \chi_{\alpha+\beta}(c_{\alpha,\beta,s,t}st)$$

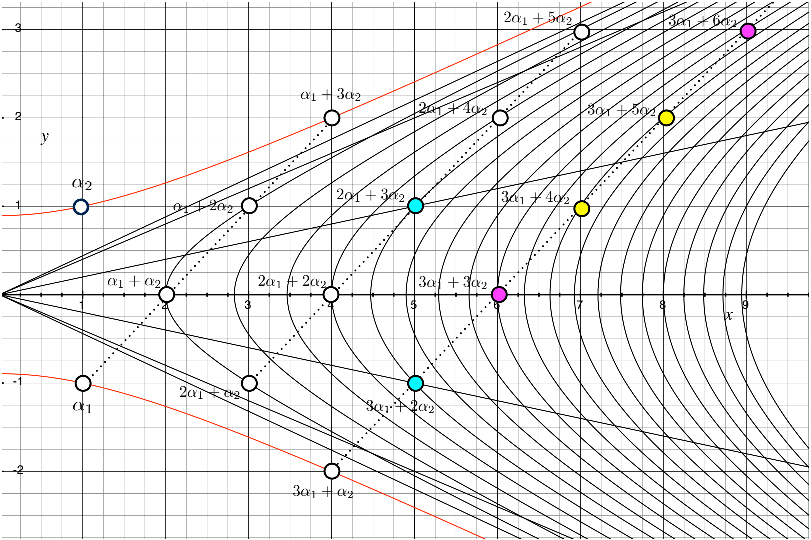
and $c_{\alpha,\beta,s,t} = n_{\alpha,\beta}$, the structure constant from $[x_\alpha, x_\beta] = n_{\alpha,\beta}x_{\alpha+\beta}$. We have $p_{\alpha,\beta} = 0$ thus

$$c_{\alpha,\beta,s,t} = s_{\alpha,\beta} = (-1)^{s_0(\alpha,\beta)}$$

where $s_0(\alpha, \beta) \in \{\pm 1\}$.

Root strings in the rank 2 Kac–Moody case

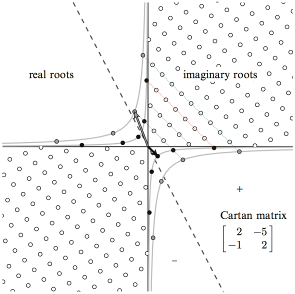
Root strings in $H(3)$:



NON–SIMPLY LACED RANK 2 KAC–MOODY CASE: PAIRS OF REAL ROOTS WHOSE SUM IS A REAL ROOT

Joint work with Matt Kownacki, Scott H. Murray and Sowmya Srinivasan

Let $\mathfrak{g}$ be the Kac–Moody algebra with (generalized) Cartan matrix $H(a,b) = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, $ab > 4$. We prove that if a and b are both greater than one, then no sum of real roots can be a real root. In the cases that a or b equal 1, we determine the pairs of short real roots whose sum is a long real root.



Structure constants and group commutators: Rank 2 Kac–Moody case

Theorem. (C., Kownacki, Murray and Srinivasan)

$A_1^{(1)}$: sum of two real roots is never a real root.

$H(m)$, $m \geq 3$: sum of two real roots is never a real root.

$H(a, b)$, $ab > 4$, $a, b \neq 1$: sum of two real roots is never a real root.

$H(a, 1)$, $a \geq 5$: the sign of $n_{\alpha,\beta}$ for every pair of real roots α, β with $\alpha + \beta$ real, is independent of every other such $n_{\gamma,\delta}$. If α and β are short roots whose sum is long, then $p_{\alpha\beta} = a - 1$, otherwise $p_{\alpha\beta} = 0$.

STRUCTURE CONSTANTS: TWISTED AFFINE CASE, $A_2^{(2)} = H(4,1)$

Suppose that $\alpha, \beta, \alpha+\beta \in \Delta^{re}$. Let $n_{\alpha,\beta} = s_{\alpha,\beta}(p_{\alpha,\beta}+1)$

If α and β are short roots whose sum is long, then $p_{\alpha,\beta} = 3$ otherwise $p_{\alpha,\beta} = 0$ (CKMS]).

Mitzman constructed Chevalley bases for the affine Kac–Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$ ([Mi]). His work includes a proof of existence of a cocycle that encodes the structure constants of $A_2^{(2)}$. This was described in more detail by Calinescu, Lepowsky and Milas ([CLM]).

Consistency of signs of structure constants (general case)

Conjecture. (C., Coulson, Kanade, McRae and Murray)
Let $\mathfrak{g}$ be a Kac–Moody algebra. Let

$$\{n_{\alpha,\beta} = s_{\alpha,\beta}(p_{\alpha,\beta}+1) \mid \alpha, \beta, \alpha+\beta \in \Delta^{re}\}$$

be a system of structure constants with signs $s_{\alpha,\beta}$ computed in the restriction to the rank two Kac–Moody subalgebra generated by $\Delta(\alpha,\beta)$ (by the formulas previously described). Then the associated Lie bracket satisfies skew symmetry and the Jacobi identity.

For $\alpha, \beta, \alpha+\beta \in \Delta^{re}$, every structure constant $n_{\alpha,\beta}$ comes from a rank 2 subsystem $\Delta(\alpha,\beta)$. Since every simple root of $\mathfrak{g}$ is contained in some rank 2 subsystem, the rank 2 subsystems generate the entire root system.

Thus the signs from rank 2 subsystems determine the signs for every possible pair of real roots whose sum is real.

Structure constants in general

Provided our conjecture is proven, this would not be a complete result: one must redefine structure constants when any of $\alpha, \beta, \alpha + \beta$ are imaginary roots, as their roots spaces are then no longer one dimensional.

We have work in progress with Coulson, Kanade and Murray that achieves this and gives an algorithm for computing structure constants consistently and efficiently over the whole Kac–Moody algebra.

Thank you!