1. Quiz - Asymptotic and perturbation methods for ordinary and partial differential equations

Remark: this is a sample of what an early assignment in the course may look like. Students should be able to solve the $\varepsilon = 0$ problem, for which answers/solutions are provided. Partial solutions for the $0 < |\varepsilon| \ll 1$ problem are also provided.

(1) Consider the perturbed eigenvalue problem

$$u'' + (\lambda + \varepsilon f(x))u = 0\,, \quad 0 < x < 1\,, \quad u(0) = u(1)\,, \quad u'(0) = u'(1)\,. \tag{1.1}$$

(a) Find all eigenvalues when $\varepsilon = 0$.
(b) Determine the first correction to the eigenvalues when $\varepsilon \ll 1$ for arbitrary $f(x)$.

(2) For the unit disk $0 \leq r \leq 1$, $0 \leq \theta < 2\pi$, consider the boundary value problem

$$\Delta u = f(r, \theta)\,, \quad 0 \leq r \leq 1\,, \quad 0 \leq \theta < 2\pi\,; \qquad u_r - \varepsilon(u - g(\theta)) = 0\,, \text{ on } r = 1\,. \tag{1.2}$$

(a) When $\varepsilon = 0$, what condition must $f(r, \theta)$ satisfy in order for (1.2) to be well-posed?
(b) Now let $0 < |\varepsilon| \ll 1$ in (1.2) with the condition in (a) not satisfied. Determine a two-term expansion for the solution u.

(c) Find the first correction term to the smallest eigenvalue for the perturbed eigenvalue problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \lambda u = 0\,, \quad 0 \leq r < 1 + \varepsilon\cos\theta\,, \tag{1.3}$$

with

$$u = 0 \text{ on } r = 1 + \varepsilon\cos\theta\,, \quad u \text{ finite as } r \to 0\,.$$

2. Solutions

(1) (a) For periodic boundary conditions, the (nonzero) eigenvalues are $\lambda_n = 4n^2\pi^2$ with $n = 1, 2, \ldots$; eigenfunctions take the form $u_n(x) = A_n \sin(\sqrt{\lambda_n}x)$ or $B_n \cos(\sqrt{\lambda_n}x)$. There are thus two eigenfunctions for each eigenvalue. Under a perturbation, we would expect this to split into two distinct eigenpairs.

(b) Let $\lambda_0 \neq 0$ be an eigenvalue of the unperturbed problem. The eigenfunction $u_0(x)$ is a linear combination $u_0(x) = A_0 \sin(2m\pi x) + B_0 \cos(2m\pi x)$ for some integer m. For small ε, we expand $\lambda = \lambda_0 + \varepsilon\lambda_1 + \cdots$ and $u = u_0 + \varepsilon u_1 + \cdots$. Substituting into (1.1) and collecting orders of ε, we have

$$\mathcal{L}u_1 \equiv u_1'' + \lambda_0 u_1 = -\lambda_1 u_0 - f(x)u_0\,, \quad u_1(0) = u_1(1)\,, \quad u_1'(0) = u_1'(1)\,. \tag{2.1}$$

The right-hand side of (2.1) must be orthogonal to the left nullspace of $\mathcal{L}$. That is,

$$\int_0^1 \sin(2m\pi x)(-\lambda_1 u_0 - f(x)u_0)\,dx = 0 \tag{2.2a}$$

$$\int_0^1 \cos(2m\pi x)(-\lambda_1 u_0 - f(x)u_0)\,dx = 0 \tag{2.2b}$$

We obtain from (2.2) a matrix eigenvalue problem for the correction term λ_1 as well as the coefficients A_0 and B_0.

(2) (a) Let Ω be the unit disk. Integrating over (1.2) over Ω, noting the pure Neumann condition on $\partial\Omega$, we have

$$\int_\Omega \Delta u = \int_{\partial\Omega} (\nabla u)\cdot\hat{n}\,ds = 0 = \int_\Omega f\,d\Omega\,,$$

so we must have $\int_\Omega f\,d\Omega = 0$.

(b) For small ε, we expand the solution $u = u_0/\varepsilon + u_1 + \varepsilon u_2 + \cdots$. Substituting into (1.2), and collecting orders of ε, we have the following sequence of equations:

$$\Delta u_0 = 0 \ \text{ in } \Omega\,, \quad u_{0r} = 0 \ \text{ on } \partial\Omega\,; \tag{2.3a}$$

$$\Delta u_1 = f \ \text{ in } \Omega\,, \quad u_{1r} = u_0 \ \text{ on } \partial\Omega\,; \tag{2.3b}$$

$$\Delta u_2 = 0 \ \text{ in } \Omega\,, \quad u_{2r} = u_1 - g(\theta) \ \text{ on } \partial\Omega\,. \tag{2.3c}$$

The solution for u_0 is simply $u_0 = \mu$ for some constant μ. The solvability condition (i.e., applying the divergence theorem to (2.3b)) yields

$$\int_\Omega f\,d\Omega = \int_\Omega u_{1r}\,ds\,,$$

which gives μ in terms of f. Next, a solvability condition on (2.3c) yields an integral condition for u_1, which when combined with (2.3b), uniquely specifies u_1 (notice that (2.3b) alone specifies u_1 only up to an arbitrary additive constant).

(3) We expand $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$ and $\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots$. Now

$$u(1+\varepsilon\cos\theta,\theta) = u_0(1,\theta) + \varepsilon[u_1(1,\theta) + u_{0r}(1,\theta)\cos\theta] + \varepsilon^2\left[u_2(1,\theta) + u_{1r}(1,\theta)\cos\theta + \frac{1}{2}u_{0rr}(1,\theta)\cos^2\theta + \cdots\right].$$

Substituting into (1.3) and collecting powers of ε, we obtain the sequence of equations on the *unperturbed* domain:

$$\Delta u_0 + \lambda_0 u_0 = 0 \ \text{ in } 0<r<1 \qquad u_0(1,\theta) = 0\,; \tag{2.4a}$$

$$\Delta u_1 + \lambda_0 u_1 = -\lambda_1 u_0 \ \text{ in } 0<r<1 \qquad u_1(1,\theta) = -u_{0r}(1,\theta)\cos\theta\,; \tag{2.4b}$$

$$\Delta u_2 + \lambda_0 u_2 = -\lambda_2 u_0 - \lambda_1 u_1 \ \text{ in } 0<r<1 \qquad u_2(1,\theta) = -u_{1r}(1,\theta)\cos\theta - \frac{1}{2}u_{0rr}(1,\theta)\cos^2\theta\,; \tag{2.4c}$$

The first eigenvalue λ_0 corresponds to a radially symmetric function, and so $u_0 = J_0(\sqrt{\lambda_0}r)$ and $\lambda_0 = z_0^2$, where $J_0(z_0) = 0$, and $J_0(x)$ is the zeroth order Bessel function of the first kind. A solvability condition for (2.4b) yields $\lambda_1 = 0$. The solution to u_1 can then be written as

$u_1(r, \theta) = f(r) \cos \theta$, where $f(r)$ is a radially symmetric function. Another solvability condition on (2.4c) yields λ_2.