Convex Optimisation: Pre-Quiz

1. Recall that the Euclidean norm (or 2-norm) of a vector $x \in \mathbb{R}^n$ is given by

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}.$$

   Show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in \mathbb{R}^n.$$

2. Compute the gradient and Hessian of the function $f\colon \mathbb{R}^n_{++} \to \mathbb{R}$ given by

$$f(x) := \frac{1}{2}\|x\|^2 - \sum_{i=1}^{n} x_i \ln(x_i).$$

3. Consider the sequence $(x_k) \subseteq \mathbb{R}$ given by

$$x_k := \frac{1}{k} + (-1)^k \quad \forall k \in \mathbb{N}.$$

   What is the value of $\liminf_{k\to\infty} x_k$ and $\limsup_{k\to\infty} x_k$?

4. Let $(x_k) \subseteq \mathbb{R}^n$ be a convergent sequence. Show that (x_k) is a Cauchy sequence.

5. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with no minima such that $\inf_{x\in\mathbb{R}} f(x)$ is finite.

Convex Optimisation: Pre-Quiz Solutions

1. Recall that the Euclidean norm (or 2-norm) of a vector $x \in \mathbb{R}^n$ is given by

$$\|x\| := \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

Show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in \mathbb{R}^n.$$

Solution: This follows by expanding the norm-squared in term of the dot-product and simplifying:

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \left(\|x\|^2 + 2\langle x, y\rangle + \|y\|^2\right) + \left(\|x\|^2 - 2\langle x, y\rangle + \|y\|^2\right) \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

2. Compute the gradient and Hessian of the function $f\colon \mathbb{R}^n_{++} \to \mathbb{R}$ given by

$$f(x) := \frac{1}{2}\|x\|^2 - \sum_{i=1}^{n} x_i \ln(x_i).$$

Solution: The entires of the gradient $\nabla f(x) \in \mathbb{R}^n$ and the Hessian $\nabla^2 f(x) \in \mathbb{R}^{n\times n}$ are given by

$$[\nabla f(x)]_i = \frac{\partial f}{\partial x_i} \quad \text{and} \quad [\nabla^2 f(x)]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j},$$

hence it suffices to compute these two partial derivatives. These are given by

$$\frac{\partial f}{\partial x_i} = x_i - \log(x_i) - 1 \quad \text{and} \quad \frac{\partial^2 f}{\partial x_j \partial x_i} = \begin{cases} 1 - \frac{1}{x_i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

3. Consider the sequence $(x_k) \subseteq \mathbb{R}$ given by

$$x_k := \frac{1}{k} + (-1)^k \quad \forall k \in \mathbb{N}.$$

What is the value of $\liminf_{k\to\infty} x_k$ and $\limsup_{k\to\infty} x_k$?

Solution: $\liminf_{k\to\infty} x_k = -1$ and $\limsup_{k\to\infty} x_k = +1$.

4. Let $(x_k) \subseteq \mathbb{R}^n$ be a convergent sequence. Show that (x_k) is a Cauchy sequence.

   Solution: Let $x \in \mathbb{R}$ denote the limit of (x_k). Using the triangle inequality, we deduce

$$\begin{aligned} \|x_n - x_k\| &= \|(x_n - x) - (x_k - x)\| \\ &\leq \|x_n - x\| + \|x_k - x\| \to 0 \quad \text{as} \quad n, k \to +\infty. \end{aligned}$$

5. Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with no minima such that $\inf_{x\in\mathbb{R}} f(x)$ is finite.

   Solution: If $f(x) = \exp(x)$, then $\inf_{x\in\mathbb{R}} f(x) = 0$ but $f(x) > 0$ for all $x \in \mathbb{R}$.