ACE 2021 **Abstract Algebra**

Pre-Enrolment Quiz

The material covered in these questions is revised at the beginning of the course and contained in the first chapter of the lecture notes. The latter are available upon request for students wanting to prepare.

Question 1.

Prove that the set X is a subset of the set Y if and only if

$$in_X^Y : X \longrightarrow Y, \quad x \longmapsto x$$

is a function (in_X^Y is the *inclusion function* of X into Y).

Question 2.

Take a non-empty set X and a function $f : X \to Y$. Show that f has

(i) a left inverse if and only if it is injective (or 1–1),

(ii) a right inverse if and only if it is surjective (or onto).

Question 3.

Show that every equivalence relation on the set X determines a unique partition of X, and conversely.

Sample Solutions

Question 1.

First suppose $X \subseteq Y$. Then both X and Y are sets, and, for every $x \in X$, $\iota_X^Y(x) = x$ is a uniquely determined element of Y. Thus ι_X^Y is a function.

For the converse, assume that ι_X^Y is a function. Then $\iota_X^Y(x) = x$ is a uniquely determined element of Y for every $x \in X$ and hence $X \subseteq Y$.

Question 2.

(i) We first suppose that $f : X \to Y$ has a left inverse, say, $g : Y \to X$.

To see that f must then be injective (that is 1–1), suppose that $f(x) = f(x')$. Then

$$x = id_X(x) = (g \circ f)(x) = g\left(f(x)\right) = g\left(f(x')\right) = (g \circ f)(x') = id_X(x') = x',$$

as required.

For the converse, suppose that $f : X \to Y$ is injective. Define $g : Y \to X$ by

$$g(y) := \begin{cases} x & \text{if } \ y = f(x) \\ x_0 & \text{otherwise, where } x_0 \text{ is some fixed element of } X. \end{cases}$$

We must first show that g so defined is, in fact a function. The only possible difficulty is that g might assign more than one element of X, say x and x', to some element y of Y. From the definition of g, this could only happen when $y \in \mathrm{im}(f)$. This, in turn, is only possible when $y = f(x) = f(x')$. But then $x = x'$, since f is injective.

That $g \circ f = id_X$ follows from the definition of g.

(ii) We first suppose that $f : X \to Y$ has a right inverse, say, $g : Y \to X$.

To see that f must be surjective, take $y \in Y$ and put $x := g(y)$. Then

$$f(x) = f\left(g(y)\right) = (f \circ g)(y) = id_Y(y) = y,$$

showing that f is, indeed surjective.

For the converse, suppose that $f : X \to Y$ is surjective. Define $g : Y \to X$ by choosing for each $y \in Y$ an element, say x_y of X with $f(x_y) = y$. This is always possible is because f is surjective.

This g is obviously a function.

Now take $y \in Y$. Then $(f \circ g)(y) = f\left(g(y)\right) = f(x_y) = y$, by the definition of g. Thus $f \circ g = id_Y$.

Question 3. First let $\sim$ be an equivalence relation on the set X.

Proposition. *Given $x, y \in X$, $x \sim y$ if and only if $[x] = [y]$.*

Proof. Take $x, y \in X$ and assume $x \sim y$. Then $z \in [x]$ implies $x \sim z$ and hence $z \sim x$ by symmetry. Thus $z \sim y$ by transitivity which in turn implies $y \sim z$, so that $z \in [y]$. This yields $[x] \subseteq [y]$. Exchanging the rôles of x and y we obtain $[y] \subseteq [x]$ and therefore $[x] = [y]$.

For the converse assume $[x] = [y]$. Reflexivity implies $x \sim x$ so that $x \in [x] = [y]$. Thus $x \sim y$. □

Reflexivity implies $x \sim x$ for every $x \in X$ and hence $x \in [x]$. Thus $[x] \neq \emptyset$ and $X = \bigcup_{x \in X} [x]$. It remains to show that, given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. So assume $[x] \cap [y] \neq \emptyset$ and take $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$ and hence $x \sim y$ by symmetry and transitivity. Hence the above proposition implies $[x] = [y]$.

Now let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a partition of the set X. Define $x \sim y$ if and only if there is a $\lambda \in \Lambda$ with $x, y \in X_\lambda$. We must show that $\sim$ is an equivalence relation.

As $\{X_\lambda \mid \lambda \in \Lambda\}$ is a partition, every $x \in X$ is contained in some X_λ and hence $x \sim x$. Clearly, $\sim$ is symmetric. Finally, assume $x \sim y$ and $y \sim z$. Then there are $\mu, \lambda \in \Lambda$ with $x, y \in X_\mu$ and $y, z \in X_\lambda$. Thus $y \in X_\mu \cap X_\lambda$ which implies $X_\mu = X_\lambda$ and thus $x \sim z$. Therefore $\sim$ is reflexive, symmetric and transitive and hence an equivalence relation.

Starting with a partition, the equivalence classes of the equivalence relation, defined as above, are the subsets of the partition. Hence the partition given by this equivalence relation is the original partition. Starting with an equivalence relation, the subsets forming the partition are the equivalence classes. Hence the equivalence relation given by this partition is the original equivalence relation. Thus there is a bijection between the set of equivalence relations on the set X and the set of partitions on the set X, showing that an equivalence relation determines a unique partition and vice versa.