Topological Groups: Pre-enrolment quiz

1. Give an ϵ-δ argument that the function $P : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$P(x, y) = x - y, \qquad x, y \in \mathbb{R},$$
is continuous.

2. Let $\mathrm{GL}(2, \mathbb{R})$ be the group of 2×2 invertible real matrices under multiplication and $(\mathbb{R} \setminus \{0\}, \times)$ be the group of non-zero real numbers under multiplication.
   (a) Explain why the map $D : \mathrm{GL}(2, \mathbb{R}) \to (\mathbb{R} \setminus \{0\}, \times)$ defined by
   $$D(A) = \det(A), \qquad A \in \mathrm{GL}(2, \mathbb{R})$$
   is a group homomorphism.
   (b) Show that $((0, \infty), \times)$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \times)$.
   (c) Explain why $D^{-1}((0, \infty))$ and $D^{-1}(\{1\})$ are subgroups of $\mathrm{GL}(2, \mathbb{R})$.
   (d) Why are they normal subgroups?
   (e) What is the index of $D^{-1}((0, \infty))$ in $\mathrm{GL}(2, \mathbb{R})$?
   (f) Is $\mathrm{GL}(2, \mathbb{R})$ connected? Explain your answer.

3. Let $\mathbb{Q}$ be the set of rational numbers and let d be the metric $d(x, y) = |x - y|$ on $\mathbb{Q}$. Explain why the interval $\{x \in \mathbb{Q} \mid |x| \leq 1\}$ is not compact.

4. Let $C_c((0, \infty))$ be the vector space of continuous, complex-valued functions on $(0, \infty)$ with compact support and define a map $\Lambda : C_c((0, \infty)) \to \mathbb{C}$ by
$$\Lambda(\phi) = \int_0^\infty \frac{\phi(x)}{|x|}\, \mathrm{d}x, \qquad \phi \in C_c((0, \infty)).$$
   (a) Explain why Λ is linear.
   (b) Show that Λ is invariant under the change of variable $x \mapsto ax$ for any $a \in (0, \infty)$.

5. Using the Orbit-Stabiliser Theorem or otherwise, find the order of the automorphism group of the cube.