1. Consider the nonlinear system

$$\frac{dx}{dt} = (y-x)(1-x-y); \quad \frac{dy}{dt} = x(2+y):$$

   (a) Determine the critical points of the system.

   (b) By considering the Jacobian matrices associated with each critical point, determine their nature (saddle, proper/improper node, spiral or centre) and their stability (stable, unstable or asymptotically stable)

2. Show the working leading to a formal solution of the initial-boundary value problem.

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0.$$

   Boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0$$

   Initial condition:

$$u(x,0) = 1 - \sin x, \quad 0 < x < \pi$$

3. Find a solution to the boundary value problem, showing the working.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi$$

$$\begin{aligned} u(0,y) &= 0, \quad u(\pi,y) = 0, \quad 0 < y < \pi \\ u(x,0) &= \sin x + \sin 3x, \quad 0 < x < \pi \\ u(x,\pi) &= 0, \quad 0 < x < \pi \end{aligned}$$

Solutions

1. Consider the nonlinear system

$$\frac{dx}{dt} = (y - x)(1 - x - y); \quad \frac{dy}{dt} = x(2 + y) :$$

   (a) Determine the critical points of the system.
   (0,0), (0,1), (3,-2), (-2,-2).

   (b) By considering the Jacobian matrices associated with each critical point, determine their nature (saddle, proper/improper node, spiral or centre) and their stability (stable, unstable or asymptotically stable)

$$J_{(0,0)} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

   which has eigenvalues $-2, 1$. Therefore it is a saddle and unstable.

$$J_{(0,1)} = \begin{pmatrix} -1 & -1 \\ 3 & 0 \end{pmatrix}$$

   which has eigenvalues $-0.5 \pm i\sqrt{2.75}$. Therefore it is a sprial sink and stable.

$$J_{(3,-2)} = \begin{pmatrix} 5 & 5 \\ 0 & 3 \end{pmatrix}$$

   which has eigenvalues $5, 3$. Therefore it is a source and unstable.

$$J_{(-2,-2)} = \begin{pmatrix} -5 & 5 \\ 0 & -2 \end{pmatrix}$$

   which has eigenvalues $-5, -2$. Therefore it is a sink and stable.

2. Show the working leading to a formal solution of the initial-boundary value problem.

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0.$$

   Boundary conditions:

$$\frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0$$

   Initial condition:

$$u(x, 0) = 1 - \sin x, \quad 0 < x < \pi$$

   Separate variables by putting $u(x, t) = X(x)T(t)$ in the PDE. This gives

$$u(x, t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-3n^2 t} \cos nx$$

From the initial conditions we obtain

$$a_0 = \frac{1}{\pi}(\pi - 2)$$

and

$$a_n = \begin{cases} 0, & n \text{ is odd} \\ \dfrac{4}{\pi(n^2-1)}, & n \text{ is even} \end{cases}$$

3. Find a solution to the boundary value problem, showing the working.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi$$

$$\begin{aligned} u(0,y) &= 0, \quad u(\pi,y) = 0, \quad 0 < y < \pi \\ u(x,0) &= \sin x + \sin 3x, \quad 0 < x < \pi \\ u(x,\pi) &= 0, \quad 0 < x < \pi \end{aligned}$$

Using separation of variables we obtain

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin nx \left(\sinh\left(n\left(y-\pi\right)\right)\right)$$

The boundary conditions gives us

$$\begin{aligned} a_1 &= \frac{1}{\sinh(\pi)} \\ a_4 &= \frac{1}{\sinh(3\pi)} \end{aligned}$$

and all other a_n are zero.