Partial Differential Equations: pre-quiz

1. Let $C[0,1] = \{f : f \text{ is continous on } [0,1]\}$ and $\|f\| = \max_{x\in[0,1]} |f(x)|$. Show that $(C[0,1], \|\cdot\|)$ is a Banach space.

2. Let l^2 be the space of square summable sequences, namely contains $\{a_k\}$ satisfying $\sum_k |a_k|^2 < \infty$. Define the linear operator

$$T(a) = (a_1/1, a_2/2, a_3/3, \cdots, a_k/k, \cdots).$$

Show that T is bounded from l^2 to l^2. What's the adjoint operator T^*? What is TT^*? What is the operator norm $\|T\|_{l^2\to l^2}$?

3. Prove that for any bounded sequence $\{a_k\}$ in l^2, there exists a subsequence $\{a_{k_j}\}$ which is weakly convergent.

4. Assume $u(x,t)$ is a smooth solution with sufficient decay in x to the following nonlinear Schrödinger equation:

$$\begin{aligned} iu_t + \Delta u =& |u|^2 u, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R} \\ u(x,0) =& u_0(x). \end{aligned} \tag{1}$$

Prove that $\|u(x,t)\|_{L^2_x} = \|u_0\|_{L^2_x}$ for any t.