Topological Groups: Pre-enrolment quiz

1. Give an ϵ-δ argument that the function $P : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$P(x, y) = x - y, \qquad x, y \in \mathbb{R},$$
is continuous.

2. Let $\mathrm{GL}(2, \mathbb{R})$ be the group of 2×2 invertible real matrices under multiplication and $(\mathbb{R} \setminus \{0\}, \times)$ be the group of non-zero real numbers under multiplication.
   (a) Explain why the map $D : \mathrm{GL}(2, \mathbb{R}) \to (\mathbb{R} \setminus \{0\}, \times)$ defined by
   $$D(A) = \det(A), \qquad A \in \mathrm{GL}(2, \mathbb{R})$$
   is a group homomorphism.
   (b) Show that $((0, \infty), \times)$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \times)$.
   (c) Explain why $D^{-1}((0, \infty))$ and $D^{-1}(\{1\})$ are subgroups of $\mathrm{GL}(2, \mathbb{R})$.
   (d) Why are they normal subgroups?
   (e) What is the index of $D^{-1}((0, \infty))$ in $\mathrm{GL}(2, \mathbb{R})$?
   (f) Is $\mathrm{GL}(2, \mathbb{R})$ connected? Explain your answer.

3. Let $\mathbb{Q}$ be the set of rational numbers and let d be the metric $d(x, y) = |x - y|$ on $\mathbb{Q}$. Explain why the interval $\{x \in \mathbb{Q} \mid |x| \leq 1\}$ is not compact.

4. Let $C_c((0, \infty))$ be the vector space of continuous, complex-valued functions on $(0, \infty)$ with compact support and define a map $\Lambda : C_c((0, \infty)) \to \mathbb{C}$ by
$$\Lambda(\phi) = \int_0^\infty \frac{\phi(x)}{|x|} \, \mathrm{d}x, \qquad \phi \in C_c((0, \infty)).$$
   (a) Explain why Λ is linear.
   (b) Show that Λ is invariant under the change of variable $x \mapsto ax$ for any $a \in (0, \infty)$.

5. Using the Orbit-Stabiliser Theorem or otherwise, find the order of the automorphism group of the cube.

Pre-enrolment quiz solutions

1. Fix $\epsilon > 0$ and let $\delta = \epsilon/2$. Then, for x_i, y_i in $\mathbb{R}$, $i = 0, 1$, if $|x_0 - x_1|$ and $|y_0 - y_1|$ are both less than δ, then

$$\begin{aligned} |P(x_0, y_0) - P(x_1, y_1)| &= |(x_0 - y_0) - (x_1 - y_1)| \\ &= |(x_0 - x_1) - (y_0 - y_1)| \\ &\leq |x_0 - x_1| + |y_0 - y_1| < \epsilon/2 + \epsilon/2. \end{aligned}$$

Hence P is continuous and is in fact uniformly continuous.
<u>Comment</u>: This argument takes the metric on $\mathbb{R}^2$ to be

$$d\big((x_0, y_0), (x_1, y_1)\big) = \max\big\{|x_0 - x_1|, |y_0 - y_1|\big\}.$$

Similar arguments apply if the metric is taken to be

$$d\big((x_0, y_0), (x_1, y_1)\big) = |x_0 - x_1| + |y_0 - y_1|$$

or

$$d\big((x_0, y_0), (x_1, y_1)\big) = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$

Since all metrics for the product topology on $\mathbb{R}^2$ are equivalent, it does not matter which one is chosen.

2. (a) $\mathrm{GL}(2, \mathbb{R})$ is the group of invertible 2×2 real matrices and a matrix A is invertible if and only if its determinant is non-zero. Hence D maps $\mathrm{GL}(2, \mathbb{R})$ to $\mathbb{R} \setminus \{0\}$ and is well-defined. The determinant satisfies that $\det(AB) = \det(A)\det(B)$ for all square matrices A and B. Hence D is a homomorphism from $\mathrm{GL}(2, \mathbb{R})$ to $(\mathbb{R} \setminus \{0\}, \times)$.
<u>Comment</u>: Since D preserves the group multiplication, it follows automatically that $D(I_2) = 1$ and $D(A^{-1}) = D(A)^{-1}$ for all $A \in GL(2, \mathbb{R})$.

(b) The product of two positive real numbers is positive, and the inverse of every positive number is positive. Hence $((0, \infty), \times)$ is closed under multiplication and the inverse map and is a subgroup of $(\mathbb{R} \setminus \{0\}, \times)$.

(c) Since $((0, \infty), \times)$ and $(\{1\}, \times)$ are subgroups of $(\mathbb{R}\setminus\{0\}, \times)$ and since the inverse image under a homomorphism of a subgroup is a subgroup, $D^{-1}((0, \infty))$ and $D^{-1}(\{1\})$ are subgroups of $\mathrm{GL}(2, \mathbb{R})$.

(d) They are normal subgroups because $(\mathbb{R} \setminus \{0\}, \times)$ is abelian and $((0, \infty), \times)$ and $(\{1\}, \times)$ are therefore normal subgroups. More explicitly, Let $A \in D^{-1}((0, \infty))$ and suppose that $B \in \mathrm{GL}(2, \mathbb{R})$. Then

$$D(BAB^{-1}) = D(B)D(A)D(B)^{-1} = D(A)$$

which is positive. Hence $BAB^{-1} \in D^{-1}((0, \infty))$.

(e) The index is 2 because $\mathbb{R}\setminus\{0\} = \{1,-1\}\times(0,\infty)$, which implies that the index of $((0,\infty),\times)$ in $(\mathbb{R}\setminus\{0\},\times)$ is 2.

(f) $\mathrm{GL}(2,\mathbb{R})$ is not connected. The determinant function is continuous and the image, $D(\mathrm{GL}(2,\mathbb{R}))$, of $\mathrm{GL}(2,\mathbb{R})$ under the determinant $\mathbb{R}\setminus\{0\}$. This set has two connected components, $(0,\infty)$ and $(-\infty,0)$.

3. For each n, let r_n be the expansion of $\sqrt{1/2}$ to n decimal places. Then $\{r_n\}_{n\in\mathbb{N}}$ is a sequence in $\{x\in\mathbb{Q} \mid |x|\leq 1\}$ that converges to $\sqrt{1/2}$ in $\mathbb{R}$. Since $\sqrt{1/2}$ is irrational, there is no subsequence that converges to a rational number. Hence $\{r_n\}_{n\in\mathbb{N}}$ has no convergent subsequence in $\{x\in\mathbb{Q} \mid |x|\leq 1\}$ and this interval is not compact.
Comment: An alternative argument is to note that $\{[-1,r_n) \mid n\in\mathbb{N}\}\cup(\sqrt{1/2},1]$ *is an open cover of* $\{x\in\mathbb{Q} \mid |x|\leq 1\}$ *that has no finite subcover.*

4. (a) Properties of the integral imply that, if $\phi_1,\phi_2\in C_c((0,\infty))$ and $b\in\mathbb{C}$, then

$$\begin{aligned}\Lambda(\phi_1+\phi_2) &= \int_0^\infty \frac{(\phi_1+\phi_2)(x)}{|x|}\,\mathrm{d}x \\ &= \int_0^\infty \frac{\phi_1(x)}{|x|}\,\mathrm{d}x + \int_0^\infty \frac{\phi_2(x)}{|x|}\,\mathrm{d}x = \Lambda(\phi_1)+\Lambda(\phi_2)\end{aligned}$$

and

$$\Lambda(b\phi_1) = \int_0^\infty \frac{b\phi_1(x)}{|x|}\,\mathrm{d}x = b\int_0^\infty \frac{\phi_1(x)}{|x|}\,\mathrm{d}x = b\Lambda(\phi_1).$$

Hence Λ is linear.

(b) For $\phi\in C_c((0,\infty))$, denote the $\phi_a(x)=\phi(ax)$. Then

$$\Lambda(\phi_a) = \int_0^\infty \frac{\phi_a(x)}{|x|}\,\mathrm{d}x = \int_0^\infty \frac{\phi(ax)}{|x|}\,\mathrm{d}x.$$

Put $u=ax$. Then $\mathrm{d}x=\frac{1}{a}\mathrm{d}u$ and

$$\int_0^\infty \frac{\phi(ax)}{|x|}\,\mathrm{d}x = \int_0^\infty \frac{\phi(u)}{\frac{1}{a}|u|}\,\frac{1}{a}\mathrm{d}u = \int_0^\infty \frac{\phi(u)}{|u|}\,\mathrm{d}u = \Lambda(\phi).$$

Hence $\Lambda(\phi_a)=\Lambda(\phi)$ and Λ is invariant under the change of variable.

5. Denote the automorphism group of the cube by G. Then G is transitive on the 8 vertices of the cube. Hence, denoting the stabiliser of the vertex v by G_v, the Orbit-Stabiliser Theorem implies that

$$|G| = 8|G_v|.$$

Next v has 3 vertices, call them r, s and t, adjacent to it and G_v is transitive on those vertices. Hence, denoting the stabiliser of v and r by $G_{v,r}$, the Orbit-Stabiliser Theorem implies that

$$|G| = 8 \times 3|G_{v,r}|.$$

Finally, $G_{v,r}$ is transitive on $\{s, t\}$ and the identity is the only automorphism that fixes v, r, s and t. Hence the Orbit-Stabiliser Theorem implies that

$$|G| = 8 \times 3 \times 2 = 48.$$

Comment: Another, not recommended, argument would be to number the vertices $1, \ldots, 8$ *and list all 48 automorphisms as permutations of* $\{1, \ldots, 8\}$.