Self-assessment: Arbitrage Pricing in Continuous Time

1. An unfair coin is tossed repeatedly. For each toss, the probability to get a head is p. Here $p \in (0,1)$ is a constant.

(1) Let X be the number of heads obtained in the first 10 tosses. Find the p.f. (probability function) of X.

(2) Let Y be the number of tails obtained before a first head is obtained. Find the p.f. of Y.

(3) Are X and Y dependent? Why or why not?

2. Let X and Y be random variables with the joint p.d.f. (probability density function)

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2+y^2), & 0<x<1,\ 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Find the marginal p.d.f. of X.

(2) Find $\mathbb{E}[X]$.

(3) Find $\mathrm{Var}[X]$.

(4) By symmetry $\mathbb{E}[X] = \mathbb{E}[Y]$ and $\mathrm{Var}[X] = \mathrm{Var}[Y]$. Now find $\mathrm{Cov}(X,Y)$ and $\rho(X,Y)$.

(5) Find the conditional expectation of Y given $X = \frac{1}{2}$.

3. Solve the following ordinary differential equation.

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0, \quad x(0) = 0,\ x'(0) = 1.$$

Solutions

1. (1)
$$f(x) = \begin{cases} \binom{10}{x} \cdot p^x (1-p)^{10-x}, & x = 0, 1, \ldots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

(2)
$$g(y) = \begin{cases} (1-p)^y \cdot p, & y = 0, 1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(3) Yes. Reason: As $X \geq 1 \implies Y \leq 9$, we have
$$\mathbb{P}(X \geq 1,\ Y \geq 10) = 0 \neq \mathbb{P}(X \geq 1) \cdot \mathbb{P}(Y \geq 10) > 0.$$

2. (1)
$$f_1(x) = \int_0^1 f(x,y)dy = \begin{cases} \frac{3}{2}x^2 + \frac{1}{2}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(2)
$$\mathbb{E}[X] = \int_0^1 x f_1(x) dx = \frac{5}{8}.$$

(3)
$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_0^1 x^2 f_1(x) dx - \frac{25}{64} = \frac{73}{960}.$$

(4)
$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = \int_0^1 \int_0^1 xy f(x,y) dx dy - \frac{5}{8} \cdot \frac{5}{8} = -\frac{1}{64}.$$
$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]} \cdot \sqrt{\mathrm{Var}[Y]}} = -\frac{15}{73}.$$

(5) For $x \in (0,1)$, the conditional p.d.f. of Y given $X = x$ is
$$g_2(y|x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2 + \frac{1}{2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then
$$\mathbb{E}\left[Y \middle| X = \frac{1}{2}\right] = \int_0^1 y \cdot g_2\left(y \middle| \frac{1}{2}\right) dy = \int_0^1 y \cdot \frac{\frac{3}{2}(\frac{1}{4} + y^2)}{\frac{3}{2} \cdot \frac{1}{4} + \frac{1}{2}} = \frac{9}{14}.$$

3. The characteristic equation is
$$y^2 - 3y + 2 = 0,$$
and the two roots are $y_1 = 1, y_2 = 2$. Then
$$x(t) = Ae^t + Be^{2t}$$
for some constants A and B. By the initial condition
$$\begin{cases} x(0) = A + B = 0, \\ x'(0) = A + 2B = 1, \end{cases} \implies \begin{cases} A = -1, \\ B = 1. \end{cases}$$

Therefore, $x(t) = -e^t + e^{2t}$.