Office Use Only

Complex Analysis diagnostic test

1. (a) Use the polar form of division to determine $z = \frac{2+2i}{\sqrt{2}-i\sqrt{2}}$ in polar form. Give the answer also in standard $a + bi$ form.

   (b) Find and plot all roots of the equation $z^3 = \frac{i}{8}$. Give the solutions also in standard $a + bi$ form.

   (c) Let $\mathcal{C}_1 = \{Re(z) = 2\}$ and $\mathcal{C}_2 = \{Im(z) = \frac{\pi}{2}\}$.

      (i) Sketch the lines $\mathcal{C}_1$ and $\mathcal{C}_2$.

      (ii) Sketch and describe the images of $\mathcal{C}_1$ and $\mathcal{C}_2$ under the exponential map $f(z) = e^z$.

2. Evaluate each of the integrals below, showing all of your working.

   (a) $\int_{\mathcal{C}} \bar{z} dz$ where $\mathcal{C}$ is the circle $\{|z| = 3\}$ traversed *anticlockwise.*

   (b) $\int_{\mathcal{C}} z^3 \cos(\frac{\pi}{2} z^4) dz$, where $\mathcal{C}$ is the curve parameterized by $z(t) = -t + (1-t)e^{\pi i/4}$ for $0 \le t \le 1$.

   (c)
$$\oint_{\mathcal{C}} (\frac{1}{z} + \sin(1/z)) dz, \tag{1}$$
   where $\mathcal{C}$ is the unit circle traversed *clockwise.*

   (d)
$$\oint_{|z+\frac{1}{2}|=2} \frac{e^{3z}}{(z+1)^2(z-\pi)^3} dz, \tag{2}$$
   where the circle is traversed *anticlockwise.*

   (e)
$$\oint_{\mathcal{C}} \frac{\sin(\pi z)}{z^2(z-\frac{1}{2})(z+\frac{1}{2})} dz, \tag{3}$$
   with $\mathcal{C}$ traversed *anticlockwise*, with

      (i) $\mathcal{C} = \{|z| = 3\}$.

      (ii) $\mathcal{C} = \{|z - \frac{1}{4}| = \frac{1}{2}\}$.

      (iii) $\mathcal{C} = \{|z+1| = \frac{3}{4}\}$.

1. a) $|2+2i| = \sqrt{2^2+2^2} = 2\sqrt{2}$ $\quad$ $\mathrm{Arg}(2+2i) = \frac{\pi}{4}$

$|\sqrt{2} - i\sqrt{2}| = \sqrt{2+2} = 2$ $\quad$ $\mathrm{Arg}(\sqrt{2} - i\sqrt{2}) = \frac{-\pi}{4}$

So $\dfrac{2+2i}{\sqrt{2}-i\sqrt{2}} = \dfrac{2\sqrt{2}\, e^{\pi i/4}}{2\, e^{-\frac{\pi i}{4}}} = \sqrt{2}\, e^{\frac{\pi i}{2}} = \sqrt{2}\, i$

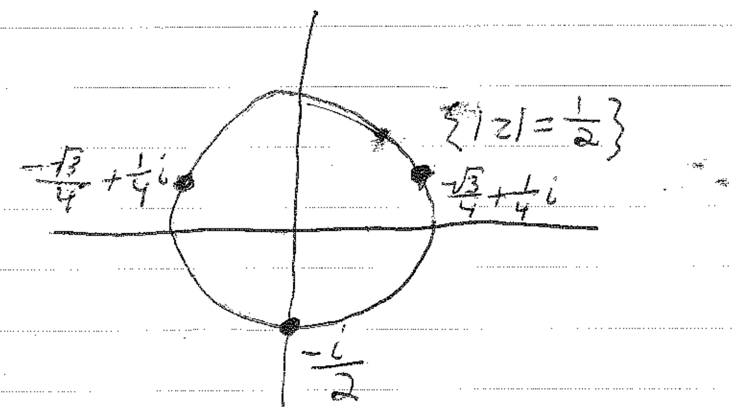
b) $\frac{i}{8} = \frac{1}{8} e^{\frac{\pi i}{2}}$, so one solution is

$\frac{1}{2} e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{4} + \frac{1}{4} i$. To find the other two,

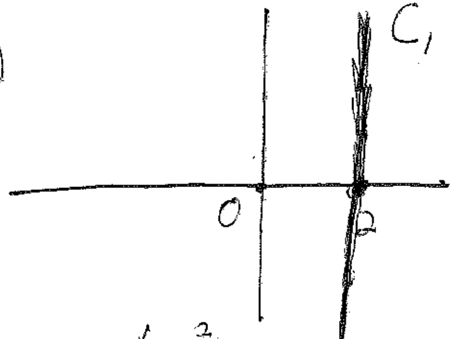
multiply by the cube roots of 1: $e^{\frac{2\pi i}{3}}$, $e^{\frac{4\pi i}{3}}$. We

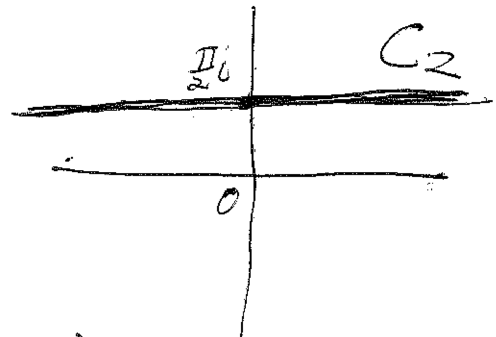
get $\frac{1}{2} e^{\frac{\pi i}{6} + \frac{2\pi i}{3}} = \frac{1}{2} e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{4} + \frac{1}{4} i$ and

$\frac{1}{2} e^{\frac{\pi i}{6} + \frac{4\pi i}{6}} = \frac{1}{2} e^{\frac{3\pi i}{2}} = \frac{-i}{2}$.

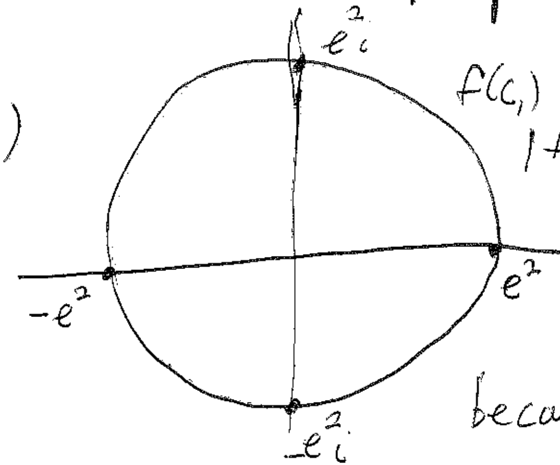


1. c)(i)

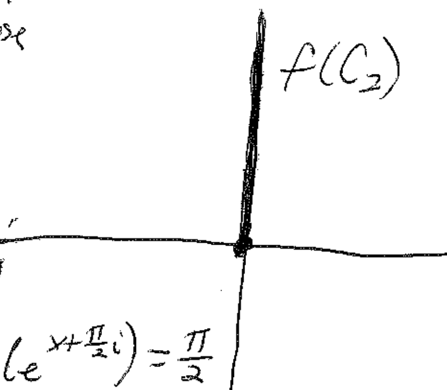




ii)



$f(C_1)$ is $\{|z| = e^2\}$, because $|f(2+yi)| = |e^{2+yi}| = e^2$



$f(C_2) = \{\arg(z) = \frac{\pi}{2}\}$,

because $\arg(x + \frac{\pi}{2}i) = \arg(e^{x+\frac{\pi}{2}i}) = \frac{\pi}{2}$

2a) $z(t) = 3e^{it}$, for $0 \le t \le 2\pi$

$z'(t) = i3e^{it}$ so

$$\int_C \bar{z}\,dz = \int_0^{2\pi} \overline{3e^{it}}\; i3e^{it}\,dt = \int_0^{2\pi} 3e^{-it}\, i3e^{it}\,dt$$

$$= 9i\int_0^{2\pi} dt = 18\pi i.$$

b) Since $\frac{d}{dz}\frac{1}{2\pi}\sin(\frac{\pi}{2}z^4) = \cos(\frac{\pi}{2}z^4)\,z^3$, and $z(0) = e^{\frac{\pi i}{4}}$, $z(1) = -1$, we have by the Fund. Thm. of Calc.

$$\int_C z^3\cos(\tfrac{\pi}{2}z^4)\,dz = \frac{1}{2\pi}\sin(\tfrac{\pi}{2}z^4)\Big|_{e^{\frac{\pi i}{4}}}^{-1}$$

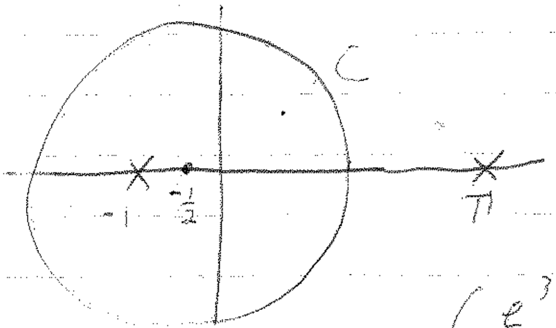
$$= \frac{1}{2\pi}\left(\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right)\right) = \frac{1}{2\pi}(1-(-1)) = \frac{1}{\pi}.$$

2(c) Since the circle is traversed clockwise, we have a -1 introduced into the formulas. $\int_C \frac{1}{z}\,dz = -2\pi i \operatorname{res}(\frac{1}{z},0)$, and $\operatorname{res}(\frac{1}{z},0) = \lim_{z\to 0} z\left(\frac{1}{z}\right) = 1$, so $\int_C \frac{1}{z}\,dz = -2\pi i$.

To find $\operatorname{res}(\sin(\frac{1}{z}),0)$, we must find the Laurent series of $\sin(\frac{1}{z})$ at $z=0$, which is $\sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{z^3 3!} + \cdots$, and we see $\operatorname{res}(\sin(\frac{1}{z}),0) = 1$ as well, so $\int_C \sin(\frac{1}{z})\,dz = -2\pi i \operatorname{res}(\sin(\frac{1}{z}),0) = -2\pi i$.

Adding these gives a final answer of $\boxed{-4\pi i}$.

2(d)



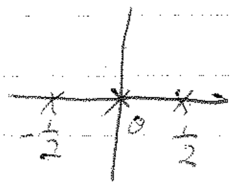
If we let $f(z) = \frac{e^{3z}}{(z-\pi)^3}$, then $f(z)$ is analytic inside C, and so, by the C.I.F.

$$\int_C \frac{e^{3z}}{(z+1)^2(z-\pi)^3}\,dz = \int_C \frac{f(z)}{(z+1)^2}\,dz = 2\pi i f'(-1)$$

$$f'(z) = \frac{(z-\pi)^3 3e^{3z} - e^{3z}3(z-\pi)^2}{(z-\pi)^6} = 3e^{3z}\frac{(z-1-\pi)}{(z-\pi)^4},$$ so

$$\int_C \frac{e^{3z}}{(z+1)^2(z-\pi)^3}\,dz = 2\pi i\, 3e^{-3}\frac{(-2-\pi)}{(-1-\pi)^4}.$$ Equivalent method using the residue theorem recieves full marks.

2 e) Let $f(z) = \frac{\sin \pi z}{z^2(z-\frac{1}{2})(z+\frac{1}{2})}$.



$\operatorname{res}(f,\frac{1}{2}) = \lim_{z\to\frac{1}{2}} (z-\frac{1}{2})f(z) = \frac{\sin \pi(\frac{1}{2})}{(\frac{1}{2})^2(\frac{1}{2}+\frac{1}{2})} = 4$

$\operatorname{res}(f,-\frac{1}{2}) = \lim_{z\to-\frac{1}{2}} (z+\frac{1}{2})f(z) = \frac{\sin \pi(-\frac{1}{2})}{(-\frac{1}{2})^2(-\frac{1}{2}-\frac{1}{2})} = 4$

$$\mathrm{res}(f,0) = \lim_{z\to 0} \frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{\sin \pi z}{(z-\frac{1}{2})(z+\frac{1}{2})}\right)$$

$$= \lim_{z\to 0} \frac{(z-\frac{1}{2})(z+\frac{1}{2})\pi\cos\pi z - \sin\pi z\left((z+\frac{1}{2})+(z-\frac{1}{2})\right)}{(z+\frac{1}{2})^2(z-\frac{1}{2})^2}$$

$$= \frac{(-\frac{1}{4})\pi(1) - 0}{(\frac{1}{2})^2(\frac{-1}{2})^2} = -4\pi$$

(i) The curve $C=\{|z|=3\}$ passes around all three singularities, so

$$\int_C f(z)\,dz = 2\pi i\left(\mathrm{res}(f,-\tfrac{1}{2}) + \mathrm{res}(f,0) + \mathrm{res}(f,\tfrac{1}{2})\right)$$
$$= 2\pi i\left(4 + -4\pi + 4\right) = 2\pi i(8-4\pi).$$

(ii) The curve $C=\{|z-\frac{1}{4}|=\frac{1}{2}\}$ passes around the singularities at $z=0,\frac{1}{2}$, but not $z=-\frac{1}{2}$, so

$$\int_C f(z)\,dz = 2\pi i\left(\mathrm{res}(f,0) + \mathrm{res}(f,\tfrac{1}{2})\right) = 2\pi i(4-4\pi).$$

(iii) The curve $C=\{|z+1|=\frac{3}{4}\}$ passes only around the singularity at $z=-\frac{1}{2}$, so

$$\int f(z)\,dz = 2\pi i\ \mathrm{res}(f,-\tfrac{1}{2}) = 8\pi i$$