THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Diagnostic Quiz

MATH4313: Functional Analysis — Semester 1, 2021

Web Page: `https://www.maths.usyd.edu.au/u/UG/HM/MATH4313/`
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1. Assume that (x_n) is a sequence in $\mathbb{R}^N$ such that $x_n \to x$. Prove that $\|x_n\| \to \|x\|$ as $n \to \infty$. Is the converse correct as well?

2. Let (z_k) be a sequence of complex numbers and assume that the series $\sum_{k=0}^{\infty} |z_k|$ converges. Show that the sequence $s_n := \sum_{k=1}^{n} z_k$ of partial sums is a Cauchy sequence in $\mathbb{C}$.

3. Let V be an inner product space over $\mathbb{C}$ with norm induced by the inner product, that is, $\|x\| = \sqrt{\langle x, x\rangle}$, where $\langle \cdot\,, \cdot\rangle$ denotes the inner product.

Given $u, v \in V$ define $p(t) := \|u - t\langle u, v\rangle\|^2$ for all $t \in \mathbb{R}$. Show that $p(t)$ is a quadratic function of $t \in \mathbb{R}$ and determine its discriminant. Hence show that $|\langle u, v\rangle| \leq \|u\|\|v\|$.

Solutions

1. We use the reversed triangle inequality to see that

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\| \to 0$$

by assumption. By the squeeze law $\big|\|x_n\| - \|x\|\big| \to 0$ as $n \to \infty$ and hence $\|x_n\| \to \|x\|$.

The converse is not true. For instance if $N = 1$ and $x_n = (-1)^n$, then x_n does not converge, bu $|x_n| = 1$ does converge.

If you do not know the reversed triangle inequality here is how to prove it. Using the triangle inequality

$$\|x_n\| = \|x_n - x + x\| \leq \|x_n - x\| + \|x\|.$$

If we rearrange we obtain

$$\|x_n\| - \|x\| \leq \|x_n - x\|.$$

Interchanging the roles of x_n and x we also have

$$\|x\| - \|x_n\| \leq \|x - x_n\| = \|x_n - x\|$$

Now combine the two inequalities.

2. Since $\sum_{k=1}^{\infty} |x_k|$ converges in $\mathbb{C}$ the sequence $a_n := \sum_{k=1}^{n} |x_k|$ converges. Hence it is a Cauchy sequence, which means that given $\varepsilon > 0$ there exists n_0 such that

$$|a_m - a_n| = \sum_{k=n+1}^{n} |z_k| < \varepsilon$$

for all $m > n > n_0$. Now, by the triangle inequality

$$|s_m - s_n| = \Big|\sum_{k=n+1}^{m} z_k\Big| \leq \sum_{k=n+1}^{m} |z_k| = |a_m - a_n| < \varepsilon$$

for all $m > n > n_0$. Hence (s_n) is a Cauchy sequence.

3. Let now $u, v \in V$ and $t \in \mathbb{R}$. Recall that inner products are conjugate linear in the second argument if the space is complex, and that $|z| = \bar{z}z$ for all $z \in \mathbb{C}$. Using the basic properties of the inner product we have

$$\begin{aligned}
0 \leq p(t) &= \|u - t\langle u, v\rangle v\|^2 = \big\langle u - t\langle u, v\rangle v, u - t\langle u, v\rangle v\big\rangle \\
&= \langle u, u\rangle - \big\langle u, t\langle u, v\rangle v\big\rangle - \big\langle t\langle u, v\rangle v, u\big\rangle + \big\langle t\langle u, v\rangle v, t\langle u, v\rangle v\big\rangle \\
&= \|u\|^2 - t\overline{\langle u, v\rangle}\langle u, v\rangle - t\langle u, v\rangle\langle v, u\rangle + t^2\overline{\langle u, v\rangle}\langle u, v\rangle\langle v, v\rangle \\
&= \|u\|^2 - 2t|\langle u, v\rangle|^2 + t^2|\langle u, v\rangle|^2\|v\|^2.
\end{aligned}$$

The above is a non-negative quadratic with real coefficients. This is only possible if its discriminant satisfies

$$|\langle u, w\rangle|^4 - |\langle u, w\rangle|^2\|u\|^2\|w\|^2 \leq 0.$$

If we rearrange the inequality the Cauchy-Schwarz inequality follows.