Student number

Semester 1 Diagnostic Quiz, 2022

School of Mathematics and Statistics

MAST90103 Random Matrix Theory

Submission deadline:

This assignment consists of 9 pages (including this page)

Question 1

Consider five matrices $A, B \in \mathbb{C}^{N\times N}$, $C \in \mathbb{C}^{N\times M}$, $D \in \mathbb{C}^{m\times N}$, and $E \in \mathbb{C}^{M\times M}$ of dimensions $N \times N$, $N \times M$, $M \times N$ and $M \times M$, respectively. Additionally, the matrix E should be invertible. Show the following three identities for the determinant.

(a) $\det[AB] = \det[A]\det[B]$ (Prove this by the Leibniz formula of the determinant),

(b) $\det\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det[A - CE^{-1}D]\det[E]$ (Prove this by using the identity (a)),

(c) $\det[\mathbf{1}_N - CD] = \det[\mathbf{1}_M - DC]$ (Prove this by using identity (b)).

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Question 2

Let $X = -X^T \in \mathbb{R}^{N\times N}$ be a real $N \times N$ antisymmetric matrix. Prove that

(a) $\det[X] = 0$ whenever N is odd and show in this case that 0 is an eigenvalue of X;

(b) all eigenvalues are imaginary and come complex conjugate pairs, meaning when λ is an eigenvalue then the complex conjugate $\lambda^* = -\lambda$ is also an eigenvalue.

(c) if $v \in \mathbb{C}^N$ is an eigenvector of X to the eigenvalue λ, then v^* is an eigenvector of X to the eigenvalue $\lambda^* = -\lambda$.

Question 3

(a) Let $a, b \in \mathbb{C}$ be two fixed complex numbers and a has a positive real part $\mathrm{Re}(a) > 0$. Prove the following integral:

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right],$$

where $\sqrt{a}$ is the principal value of the square root of a mean it has a branch cut along the negative real line.

(b) Let $A \in \mathbb{R}^{3\times 3}$ be an invertible 3×3 real matrix. Compute the Gaussian integral

$$I(A) = \int_{\mathbb{R}^3} \exp[-x^T A^T A x] d^3x$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is a three dimensional column vector and the volume element is $d^3x = dx_1 dx_2 dx_3$.

End of Assignment

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Semester 1 Diagnostic Quiz, 2022

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Question 1

Consider five matrices $A, B \in \mathbb{C}^{N\times N}$, $C \in \mathbb{C}^{N\times M}$, $D \in \mathbb{C}^{m\times N}$, and $E \in \mathbb{C}^{M\times M}$ of dimensions $N \times N$, $N \times M$, $M \times N$ and $M \times M$, respectively. Additionally, the matrix E should be invertible. Show the following three identities for the determinant.

(a) $\det[AB] = \det[A]\det[B]$ (Prove this by the Leibniz formula of the determinant),

(b) $\det\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det[A - CE^{-1}D]\det[E]$ (Prove this by using the identity (a)),

(c) $\det[\mathbf{1}_N - CD] = \det[\mathbf{1}_M - DC]$ (Prove this by using identity (b)).

(a) The Leibniz formula of the determinant of a matrix A states

$$\det[A] = \sum_{\omega\in\mathbb{S}_N} \mathrm{sign}(\omega) \prod_{j=1}^{N} A_{j\sigma(j)}$$

with $\mathbb{S}_N$ the symmetric group comprising all permutations of N elements and $\mathrm{sign}(\omega)$ is $+1$ when ω is an even permutation and -1 when it is an odd one. Since $\{AB\}_{ab} = \sum_{c=1}^{N} A_{ac}B_{cb}$ it holds

$$\begin{aligned}
\det[AB] =& \sum_{\omega\in\mathbb{S}_N} \mathrm{sign}(\omega) \prod_{j=1}^{N} \left(\sum_{c_j=1}^{N} A_{jc_j} B_{c_j\sigma(j)} \right) \\
=& \sum_{c_1=1}^{N} \cdots \sum_{c_N=1}^{N} \sum_{\omega\in\mathbb{S}_N} \mathrm{sign}(\omega) \prod_{j=1}^{N} A_{jc_j} B_{c_j\sigma(j)} \\
=& \sum_{c_1=1}^{N} \cdots \sum_{c_N=1}^{N} \left(\prod_{j=1}^{N} A_{jc_j} \right) \sum_{\omega\in\mathbb{S}_N} \mathrm{sign}(\omega) \prod_{j=1}^{N} B_{c_j\sigma(j)} \\
=& \sum_{c_1=1}^{N} \cdots \sum_{c_N=1}^{N} \left(\prod_{j=1}^{N} A_{jc_j} \right) \det \begin{bmatrix} B_{c_1 1} & \dots & B_{c_1 N} \\ \vdots & & \vdots \\ B_{c_N 1} & \dots & B_{c_N N} \end{bmatrix}.
\end{aligned}$$

The determinant in B vanishes whenever two c_j agree. Hence, $(c_1, \dots, c_N)$ must be a permutation $\sigma \in \mathbb{S}_N$ of $(1, \dots, N)$. This means

$$\det[AB] = \sum_{\sigma\in\mathbb{S}_N} \left(\prod_{j=1}^{N} A_{jc_j} \right) \det \begin{bmatrix} B_{\sigma(1)1} & \dots & B_{\sigma(1)N} \\ \vdots & & \vdots \\ B_{\sigma(N)1} & \dots & B_{\sigma(N)N} \end{bmatrix}.$$

Reordering the rows yields the sign $\mathrm{sign}(\sigma)$ which gives us the desired result

$$\begin{aligned}
\det[AB] =& \sum_{\sigma\in\mathbb{S}_N} \left(\prod_{j=1}^{N} A_{jc_j} \right) \mathrm{sign}(\sigma) \det \begin{bmatrix} B_{11} & \dots & B_{1N} \\ \vdots & & \vdots \\ B_{N1} & \dots & B_{NN} \end{bmatrix} \\
=& \det[B] \sum_{\sigma\in\mathbb{S}_N} \left(\prod_{j=1}^{N} A_{jc_j} \right) \mathrm{sign}(\sigma) = \det[A]\det[B].
\end{aligned}$$

(b) We write the matrix in the determinant as the product of two matrices

$$\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \begin{bmatrix} A - CE^{-1}D & CE^{-1} \\ 0 & \mathbf{1}_M \end{bmatrix} \begin{bmatrix} \mathbf{1}_N & 0 \\ D & E \end{bmatrix}.$$

Applying identity (a) we have

$$\det \begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det \begin{bmatrix} A - CE^{-1}D & CE^{-1} \\ 0 & \mathbf{1}_M \end{bmatrix} \det \begin{bmatrix} \mathbf{1}_N & 0 \\ D & E \end{bmatrix}.$$

The first term is equal to $\det[A - CE^{-1}D]$ and the second term is equal to $\det[E]$ due to their triangular form.

(c) Setting $A = \mathbf{1}_N$ and $E = \mathbf{1}_M$ in identity (b), we have

$$\det \begin{bmatrix} \mathbf{1}_N & C \\ D & \mathbf{1}_M \end{bmatrix} = \det[\mathbf{1}_N - CD].$$

However, we can also write

$$\begin{bmatrix} \mathbf{1}_N & C \\ D & \mathbf{1}_M \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_M & 0 \end{bmatrix} \begin{bmatrix} \mathbf{1}_M & D \\ C & \mathbf{1}_N \end{bmatrix} \begin{bmatrix} 0 & \mathbf{1}_M \\ \mathbf{1}_N & 0 \end{bmatrix}.$$

Applying, once again identity (a) and then identity (b) with using $\det[B] = \det[B^T]$ and that the first and second matrix are orthogonal and transpose to each other we find

$$\begin{aligned} \det \begin{bmatrix} \mathbf{1}_N & C \\ D & \mathbf{1}_M \end{bmatrix} &= \det \begin{bmatrix} 0 & \mathbf{1}_N \\ \mathbf{1}_M & 0 \end{bmatrix} \det \begin{bmatrix} \mathbf{1}_M & D \\ C & \mathbf{1}_N \end{bmatrix} \det \begin{bmatrix} 0 & \mathbf{1}_M \\ \mathbf{1}_N & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} \mathbf{1}_M & D \\ C & \mathbf{1}_N \end{bmatrix} \\ &= \det[\mathbf{1}_M - DC]. \end{aligned}$$

In the last step we have anew used identity (b).

Question 2

Let $X = -X^T \in \mathbb{R}^{N\times N}$ be a real $N \times N$ antisymmetric matrix. Prove that

(a) $\det[X] = 0$ whenever N is odd and show in this case that 0 is an eigenvalue of X;

(b) all eigenvalues are imaginary and come complex conjugate pairs, meaning when λ is an eigenvalue then the complex conjugate $\lambda^* = -\lambda$ is also an eigenvalue.

(c) if $v \in \mathbb{C}^N$ is an eigenvector of X to the eigenvalue λ, then v^* is an eigenvector of X to the eigenvalue $\lambda^* = -\lambda$.

(a) It holds

$$\det[X] = \det[X^T] = \det[-X] = (-1)^N \det[X].$$

Thence, for odd N we obtain $\det[X] = -\det[X]$ which means $\det[X] = 0$. An eigenvalue λ of X is a zero of the characteristic polynomial $p(\lambda) = \det[\lambda \mathbf{1}_N - X]$ meaning we have to find $\lambda \in \mathbb{C}$ that satisfies

$$p(\lambda) = 0.$$

We have seen that $p(0) = 0$ when N is odd. Therefore, $\lambda = 0$ is an eigenvalue of X.

(b) The matrix $H = iX$ is Hermitian, i.e., the Hermitian conjugate $H^\dagger = -iX^T = iX = H$. This implies that all eigenvalues of H are real and, equivalently, all eigenvalues of X are imaginary. Due to the antisymmetry the characteristic polynomial satisfies the following property:

$$\begin{aligned} p(\lambda) &= \det[\lambda \mathbf{1}_N - X] = \det[(\lambda \mathbf{1}_N - X)^T] \\ &= \det[\lambda \mathbf{1}_N + X] = (-1)^N \det[-\lambda \mathbf{1}_N - X] = (-1)^N p(-\lambda). \end{aligned}$$

Therefore, when $\lambda \in \mathbb{C}$ is an eigenvalue it satisfies $p(\lambda) = 0$. This implies $p(-\lambda) = (-1)^N p(\lambda) = 0$ or, equivalently, $-\lambda = \lambda^*$ (because it is imaginary) is a zero of the characteristic polynomial and, hence, an eigenvalue of X, too.

(c) Let $v \in \mathbb{C}^N$ be an eigenvector of X corresponding to the eigenvalue X, then it satisfies

$$Xv = \lambda v.$$

As X is real the complex conjugate of this equation yields

$$Xv^* = (Xv)^* = (\lambda v)^* = \lambda^* v^*$$

meaning V^* is an eigenvector of X corresponding to the eigenvalue $\lambda^* = -\lambda$.

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(a) Let $a, b \in \mathbb{C}$ be two fixed complex numbers and a has a positive real part $\mathrm{Re}(a) > 0$. Prove the following integral:

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right],$$

where $\sqrt{a}$ is the principal value of the square root of a mean it has a branch cut along the negative real line.

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$$I(A) = \int_{\mathbb{R}^3} \exp[-x^T A^T A x] d^3x$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is a three dimensional column vector and the volume element is $d^3x = dx_1 dx_2 dx_3$.

(a) We first complete the squares $-ax^2 + 2bx = -a(x-b)^2 + b^2/a$ and, then, understand the integral as a contour integral of a holomorphic function along the contour $\mathcal{C} = \mathbb{R}$,

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \exp\left[\frac{b^2}{a}\right] \int_{\mathcal{C}} \exp\left[-a(z-b)^2\right] dz.$$

Next we shift the contour $z \to z + b$ which is allowed as the integrand is holomorphic and it drops of to infinity faster than $1/z^2$,

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \exp\left[\frac{b^2}{a}\right] \int_{\mathcal{C}} \exp\left[-az^2\right] dz.$$

The same also allows us to rotate and stretch the contour $z \to z/\sqrt{a}$ because $\mathrm{Re}(a) > 0$. Indeed the integrable domain at infinity for the Gaussian is when the argument of z lies in $(-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4)$. Since we start with $\arg(z) \in \{0, \pi\}$ and $\arg(a) \in (-\pi/2, \pi/2)$, it holds that a rotation from z to $z/\sqrt{a} = |z/\sqrt{a}| \exp[i(\arg(z) - \arg(a)/2)]$ crosses an argument in the integrable domain $(-\pi/4, \pi/4) \cup (3\pi/4, 5\pi/4)$. Therefore we end up with

$$\begin{aligned}\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx =& \frac{1}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right] \int_{\mathcal{C}} \exp\left[-z^2\right] dz \\ =& \frac{1}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right] \int_{\mathbb{R}} \exp\left[-x^2\right] dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right].\end{aligned}$$

(b) We substitute $y = Ax \in \mathbb{R}^3$ since A is a bijective map from $\mathbb{R}^3$ to $\mathbb{R}^3$. This means it holds $x = A^{-1}y$ and the Jacobian is given by

$$\begin{aligned}\det\begin{bmatrix} \partial_{x_1} y_1 & \partial_{x_1} y_2 & \partial_{x_1} y_3 \\ \partial_{x_2} y_1 & \partial_{x_2} y_2 & \partial_{x_2} y_3 \\ \partial_{x_3} y_1 & \partial_{x_3} y_2 & \partial_{x_3} y_3 \end{bmatrix} =& \det\begin{bmatrix} \{A^{-1}\}_{11} & \{A^{-1}\}_{12} & \{A^{-1}\}_{13} \\ \{A^{-1}\}_{21} & \{A^{-1}\}_{22} & \{A^{-1}\}_{23} \\ \{A^{-1}\}_{31} & \{A^{-1}\}_{32} & \{A^{-1}\}_{33} \end{bmatrix} \\ =& \det[A^{-1}] = \frac{1}{\det[A]}.\end{aligned}$$

This means

$$\begin{aligned}\int_{\mathbb{R}^3} \exp[-x^T A^T A x] d^3x =& \int_{\mathbb{R}^3} \exp[-y^T y] \frac{d^3 y}{|\det[A]|} \\ =& \frac{\int_{-\infty}^{\infty} e^{-y^2} dy_1)^3}{|\det[A]|} = \frac{\pi^{3/2}}{|\det[A]|}.\end{aligned}$$

End of Assignment