C^*-algebras diagnostic quiz

(1) Let (X, τ) be a topological space. Suppose that $f : X \to \mathbb{C}$ is continuous. Suppose that $x \in X$ and that $x_n \to x$ in X. Prove that $f(x_n) \to f(x)$. Is the converse true? That is, if $f(x_n) \to f(x)$ for every convergent sequence x_n in x, does it follow that f is continuous?

(2) Let $\big(H, (\cdot \mid \cdot)\big)$ be a complex Hilbert space (with inner-product conjugate-linear in the second variable), and fix $h, k \in H$. Prove that the formula $\Theta_{h,k}(l) = (l \mid k)h$ defines a linear $\Theta_{h,k} : H \to H$ that is bounded in the sense that there exists $M > 0$ such that $\|\Theta_{h,k}(l)\| \leq M\|l\|$ for all $l \in H$.

(3) Let (x_n) be a Cauchy sequence of real numbers. Prove that $\liminf x_n = \limsup x_n$ and that $x_n \to \liminf x_n$.

Solutions

(1) Fix $\varepsilon > 0$. The open ball $U := B(f(x), \varepsilon)$ is open. Since f is continuous, the preimage $f^{-1}(U)$ is open in X. By definition, since $x_n \to x$, there exists N large enough so that $x_n \in f^{-1}(U)$ for all $n \geq N$. That is $f(x_n) \in U$ for all $n \geq N$, and so $n \geq N$ implies $|f(x_n) - f(x)| < \varepsilon$.

The converse is false (it's okay if you just know this fact but have not seen/can't think of an example). For example, consider the space $X := \mathbb{R}$ with the topology $\tau = \{\emptyset\} \cup \{\mathbb{R} \setminus S : S \text{ is countable}\}$. If (x_n) is a sequence in $\mathbb{R}$ and $x_n \to x$ with respect to τ, then the set $S = \{x_n : n \in \mathbb{N}\} \setminus \{x\}$ is countable, and so $U := \mathbb{R} \setminus S$ is an open set containing x. Since $x_n \to x$, we must have $x_n \in U$ for large n, and so x_n is eventually constant. That is the only sequences that converge are eventually constant sequences. It follows that every function from $\mathbb{R}$ to $\mathbb{C}$ has the property that if $x_n \to x$ in $\mathbb{R}$ with respect to τ, then $f(x_n) \to f(x)$. However, the function $f : X \to \mathbb{C}$ given by $f(x) = x$ is not continuous: the preimage $f^{-1}(B(0; 1))$ of the open ball of radius 1 around 0 is $(-\infty, 1] \cup [1, \infty)$ which is neither empty nor the complement of a countable set and therefore not open in (X, τ).

(2) We calculate $\Theta_{h,k}(\alpha l + l') = (\alpha l + l' \mid k)h = \alpha(l \mid k)h + (l' \mid k)h$, so $\Theta_{h,k}$ is linear. We have $\|\Theta_{h,k}(l)\| = |(l \mid k)|\|h\| \leq \|l\|\|k\|\|k\|$ by the Cauchy–Schwarz inequality, so $M = \|h\|\|k\|$ is a bound for $\Theta_{h,k}$.

(3) For the first statement it suffices to show that for every $\varepsilon > 0$, we have $|\limsup x_n - \liminf x_n| \leq \varepsilon$. So fix $\varepsilon > 0$. Since (x_n) is Cauchy there exists N such that $|x_m - x_n| < \varepsilon/3$ for all $m, n \geq N$. Since $(\sup\{x_i : i \geq n\})_n$ is a nonincreasing sequence converging to $\limsup x_n$, we have $\limsup x_n \leq \sup\{x_i : i \geq N\}$ and $\liminf x_n \geq \inf\{x_i : i \geq N\}$. Fix $m, n \geq N$ such that $\sup\{x_i : i \geq M\} \leq x_m + \varepsilon/3$ and $\inf\{x_i : i \geq M\} \geq x_n - \varepsilon/3$. Since $m, n \geq N$ we have $|x_m - x_n| < \varepsilon/3$. Since $\sup\{x_i : i \geq n\} \geq \inf\{x_i : i \geq n\}$ for all n, we have $\limsup x_n \geq \liminf x_n$, and hence

$$\begin{aligned} |\limsup x_n - \liminf x_n| &= \limsup x_n - \liminf x_n < \sup\{x_i : i \geq N\} - \inf\{x_i : i \geq N\} \\ &< x_m + \varepsilon/3 - (x_n - \varepsilon/3) \leq |x_m - x_n| + 2\varepsilon/3 = \varepsilon. \end{aligned}$$

Now since $\inf\{x_i : i \geq n\} \leq x_n \leq \sup\{x_i : i \geq n\}$ for all n, and since $\lim_n \inf\{x_i : i \geq n\} = \liminf x_n = \limsup x_n = \lim_n \sup\{x_i : i \geq n\}$, the sandwich lemma implies that $x_n \to \liminf x_n$.