THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Diagnostic Quiz

MATH4061: Metric Spaces — Semester 1, 2022

Web Page: `https://www.sydney.edu.au/units/MATH4061/2022-S1C-ND-RE`
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1. Assume that (x_n) is a sequence in $\mathbb{C}$ such that $x_n \to x$. Prove that $|x_n| \to |x|$ as $n \to \infty$. Is the converse correct as well?

2. Suppose that $A \subseteq R$ is non-empty and bounded from above. Show that for an upper bound M of A the following statements are equivalent:

(i) $M = \sup(A)$;

(ii) $M \in R$ is such that for A such that for every $\varepsilon > 0$ there exists $a \in A$ such that $a > M - \varepsilon$.

Here $\sup(A)$ is the supremum (or least upper bound) of A.

3. (a) Determine whether or not the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converges in $\mathbb{R}$.

(b) Determine the spectral radius of the power series

$$\sum_{k=1}^{\infty} k! z^{k!}$$

in $\mathbb{C}$.

4. Let V be an inner product space over $\mathbb{C}$ with norm induced by the inner product, that is, $\|x\| = \sqrt{\langle x, x\rangle}$, where $\langle \cdot\,, \cdot \rangle$ denotes the inner product.

Given $u, v \in V$ define $p(t) := \|u - t\langle u, v\rangle\|^2$ for all $t \in \mathbb{R}$. Show that $p(t)$ is a quadratic function of $t \in \mathbb{R}$ and determine its discriminant. Hence show that $|\langle u, v\rangle| \leq \|u\|\|v\|$.

Solutions

1. We use the reversed triangle inequality to see that

$$\big||x_n| - |x|\big| \le |x_n - x| \to 0$$

by assumption. By the squeeze law $\big||x_n| - |x|\big| \to 0$ as $n \to \infty$ and hence $|x_n| \to |x|$.

The converse is not true. For instance if $N = 1$ and $x_n = (-1)^n$, then x_n does not converge, bu $|x_n| = 1$ does converge.

If you do not know the reversed triangle inequality here is how to prove it. Using the triangle inequality

$$|x_n| = |x_n - x + x| \le |x_n - x| + |x|.$$

If we rearrange we obtain

$$|x_n| - |x| \le |x_n - x|.$$

Interchanging the roles of x_n and x we also have

$$|x| - |x_n| \le |x - x_n| = |x_n - x|$$

Now combine the two inequalities.

2. (a) We use the ratio test to check for convergence. If a_n is the n-th term in the series We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1})} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow{n\to\infty} \frac{1}{e} < 1$$

Hence the series converges.

(b) By the Cauchy-Hadamard theorem, the radius of convergence is

$$\varrho = \frac{1}{\lim_{k\to\infty} \sqrt[k!]{k!}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{1} = 1.$$

3. (i) $\Longrightarrow$ (ii): We give a proof by contrapositive. If there exists $\varepsilon_0 > 0$ such that $M - \varepsilon_0 > a$ for all $a \in A$, then $M - \varepsilon_0$ is an upper bound with $M > M - \varepsilon_0$. Hence M is not the supremum of A.

(ii) $\Longrightarrow$ (i): Since by assumption M is an upper bound, we have to show that every upper bound m satisfies $M \le m$. We again use a contrapositive and assume that $m < M$ and $\varepsilon := M - m$. By assumption there exists $a \in A$ with $a > M - \varepsilon = M - (M - m) = m$. Hence m is not an upper bound for A.

4. Let now $u, v \in V$ and $t \in \mathbb{R}$. Recall that inner products are conjugate linear in the second argument if the space is complex, and that $|z| = \bar{z}z$ for all $z \in \mathbb{C}$. Using the basic properties of the inner product we have

$$\begin{aligned}
0 \le p(t) &= \|u - t\langle u,v\rangle v\|^2 = \big\langle u - t\langle u,v\rangle v, u - t\langle u,v\rangle v\big\rangle \\
&= \langle u,u\rangle - \big\langle u, t\langle u,v\rangle v\big\rangle - \big\langle t\langle u,v\rangle v, u\big\rangle + \big\langle t\langle u,v\rangle v, t\langle u,v\rangle v\big\rangle \\
&= \|u\|^2 - t\overline{\langle u,v\rangle}\langle u,v\rangle - t\langle u,v\rangle\langle v,u\rangle + t^2\overline{\langle u,v\rangle}\langle u,v\rangle\langle v,v\rangle \\
&= \|u\|^2 - 2t|\langle u,v\rangle|^2 + t^2|\langle u,v\rangle|^2\|v\|^2.
\end{aligned}$$

The above is a non-negative quadratic with real coefficients. This is only possible if its discriminant satisfies

$$|\langle u, w\rangle|^4 - |\langle u, w\rangle|^2 \|u\|^2 \|w\|^2 \leq 0.$$

If we rearrange the inequality the Cauchy-Schwarz inequality follows.