The Universty of Sydney
School of Mathematics and Statistics

Diagnostic Quiz
MATH4512 Stochastic Analysis

Semester 2, 2023
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1. Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution.

(a) Is it true that

$$\mathbb{E}\left(\frac{X}{X+Y}\right) = \mathbb{E}\left(\frac{Y}{X+Y}\right)?$$

(b) Suppose additionally that $X \leq Y$. Is that true that $X = Y$ a.s.?

If the answer is "yes" prove the claim, if the answer is "no" provide a counterexample.

2. Let $Y, X, X_1, X_2, ...$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

(a) If for $\varepsilon_n \searrow 0$ it holds that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon_n) < \infty$, then $X_n \to X$ a.s.

(b) If $X_n \xrightarrow{\mathbb{P}} X$ and $X_n \xrightarrow{\mathbb{P}} Y$, then $X = Y$ a.s.

3. A random variable X has exponential distribution with parameter $\alpha > 0$ if its distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Let X_1 and X_2 be two independent random variables with exponential distribution with parameters $\alpha > 0$ and $\beta > 0$ respectively. Find the distribution function and probability density function of $\min\{X_1, X_2\}$. Compute the probability $\mathbb{P}(X_1 < X_2)$.

(b) Random variables X and Y are independent and have exponential distribution with the same parameter α. Find the distribution of $X - Y$.

4. Show that two random variables X and Y on $(\Omega, \mathcal{F}, \mathbb{P})$ are equal a.s. if and only if

$$\mathbb{E}\Big(h(X)f(Y)\Big) = \mathbb{E}\Big(h(X)f(X)\Big)$$

for any $h, f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ bounded measurable functions

5. Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$ and $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Let $\mathcal{Y}$ be a sub σ algebra and X be a random variable given by:

$$\mathcal{Y} = \sigma([0, \frac{1}{4}], [\frac{1}{4}, 1]), \quad \text{and} \quad X(\omega) = \sqrt{\omega}.$$

Compute the conditional expectation $\mathbb{E}(X|\mathcal{Y})$.

6. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a filtration on that space. Suppose that τ_1 and τ_2 are two stopping times.

(a) Show that $\tau_1 + \tau_2$ is a stopping time.

(b) Assume that $\tau_2 \geq \tau_1$. Is it true that $\tau_2 - \tau_1$ is a stopping time? Either prove that it is or provide an example showing that it is not.